Lecture-18: Tractable Random Processes

1 Examples of Tractable Stochastic Processes

In general, it is very difficult to characterize a stochastic process completely in terms of its finite dimensional distribution. However, we have listed few analytically tractable examples below, where we can completely characterize the stochastic process.

1.1 Independent and identically distributed processes

Let $\{X_t : t \in T\}$ be an independent and identically distributed (iid) random process, with a common distribution $F(x)$. Then, the finite dimensional distribution for this process for any finite $S \subseteq T$ can be written as

$$F_S(x) = P\left(\{X_s(\omega) \leq x_s, s \in S\}\right) = \prod_{s \in S} F(x_s).$$

It's easy to verify that the first and the second moments are independent of time indices. Since $X_t = X_0$ in distribution,

$$m_X = \mathbb{E}X_0, \qquad R_X = \mathbb{E}X_0^2, \qquad C_X = \mathrm{Var}(X_0).$$

1.2 Stationary processes

A stochastic process X_t is **stationary** if all finite dimensional distributions are shift invariant, that is for finite $S \subseteq T$ and $t > 0$, we have

$$F_S(x_S) = P(\{X_s(\omega) \leq x_s, s \in S\}) = P(\{X_{s+t}(\omega) \leq x_s, s \in S\}) = F_{t+S}(x_S).$$

In particular, all the moments are shift invariant. Since $X_t = X_0$ and $(X_t, X_s) = (X_{t-s}, X_0)$ in distribution, we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t-s, 0) = \mathbb{E}X_{t-s}X_0, \qquad C_X(t-s, 0) = R_X(t-s, 0) - m_X^2.$$

1.3 Markov processes

A stochastic process X_t is **Markov** if conditioned on the present state, future is independent of the past. That is, for any ordered index set T containing any two indices $u > t$, we have

$$P(\{X_u \leq x_u\} | \mathcal{F}_t) = P(\{X_u \leq x_u\} | \sigma(X_t)).$$

We will study this process in detail in coming lectures.

1.4 Lévy processes

A right continuous with left limits stochastic process $X = (X_t \in \mathbb{R} : t \in T \subseteq \mathbb{R}_+)$ with $X_0 = 0$ almost surely, is a **Lévy process** if the following conditions hold.

(L1) The increments are independent. For any $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L2) The increments are stationary. For any $s < t$, $X_t - X_s$, is equal in distribution to X_{t-s}.

(L3) Continuous in probability. For any $\varepsilon > 0$ and $t \geq 0$ it holds that $\lim_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$.

Example 1.1. Two examples of Lévy processes are Poisson process and Wiener process. The distribution of Poisson process at time t is Poisson with rate λt and the distribution of Wiener process at time t is zero mean Gaussian with variance t.

Theorem 1.2. *A Lévy process has infinite divisibility. That is, for all $n \in \mathbb{N}$*

$$\mathbb{E}e^{\theta X_t} = \left(\mathbb{E}e^{\theta X_{t/n}}\right)^n.$$

Further, if the process has finite moments $\mu_n(t) = \mathbb{E}X_t^n$ then the following Binomial identity holds

$$\mu_n(t+s) = \sum_{k=0}^{n} \binom{n}{k} \mu_k(t)\mu_{n-k}(s).$$

Proof. The first equality follows from the independent and stationary increment property of the process, and the fact that we can write

$$X_t = \sum_{k=1}^{n} X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}}.$$

Second property also follows from the the independent and stationary increment property of the process, and the fact that we can write

$$X_{t+s}^n = (X_t + X_{t+s} - X_t)^n = \sum_{k=0}^{n} \binom{n}{k} X_t^k (X_{t+s} - X_t)^{n-k}.$$

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