Lecture-19: Bernoulli Processes

1 Construction of Probability Space

Consider an experiment, where an infinite sequence of trials is conducted. Each trial has two possible outcomes, success or failure, denoted by S and F respectively. Any outcome of the experiment is an infinite sequence of successes and failures, e.g.

$$\omega = (S, F, F, S, F, S, \dots).$$

The collection of all possible outcomes of this experiment will be our sample space $\Omega = \{S, F\}^{\mathbb{N}}$. The ith projection of an outcome sequence $\omega \in \Omega$ is denoted by $\omega_i \in \{S, F\}$. We consider a σ-algebra $\mathcal{F}$ on this space generated by all finite subsets of the sample space Ω.

$$\mathcal{F} = \sigma(\{\omega \in \Omega : \omega_i \in \{S, F\}, \forall i \in I \subset \mathbb{N} \text{ for finite } I\}).$$

We further assume that each trial is independent and identically distributed, with common distribution of a single trial

$$P\{\omega_i = S\} = p, \qquad P\{\omega_i = F\} = q \triangleq 1 - p.$$

This assumption completely characterizes the probability measure over all elements of the σ-algebra $\mathcal{F}$. For $a \in \mathcal{F}$ and the number of successes $n = |\{i \in I : a_i = S\}|$ in I,

$$P(a) = \prod_{i \in I} \mathbb{E}1\{\omega_i = a_i\} = \prod_{i \in I : \omega_i = S} \mathbb{E}1\{\omega_i = S\} \prod_{i \in I : \omega_i = F} \mathbb{E}1\{\omega_i = F\} = p^n q^{|I| - n}.$$

Hence, we have completely characterized the probability space $(\Omega, \mathcal{F}, P)$. Further, we define a discrete random process $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ such that

$$X_n(\omega) = 1\{\omega_n = S\}.$$

Since, each trial of the experiment is iid, so is each X_n.

2 Bernoulli Processes

For a probability space $(\Omega, \mathcal{F}, P)$, a discrete process $X = \{X_n(\omega) : n \in \mathbb{N}\}$ taking value in $\{0, 1\}^{\mathbb{N}}$ is a **Bernoulli Process** with success probability $p = \mathbb{E}X_n$ if $\{X_n : n \in \mathbb{N}\}$ are iid with common distribution $P\{X_n = 1\} = p$ and $P\{X_n = 0\} = q$.

Example 2.1. Examples of Bernoulli processes.

i. For products manufactured in an assembly line, X_n indicates the event of nth product being defective.

ii. At a fork on the road, X_n indicates the event of nth vehicle electing to go left on the fork.

Let $n(x_S) \triangleq |\{i \in S : 0 \leq x_i < 1\}|$, then the finite dimensional distribution of $X(\omega)$ is given by

$$F_S(x_S) = \prod_{i \in S} P\{X_i \leq x_i\} = q^{n(x_S)}.$$

The mean, correlation, and covariance functions are given by

$$m_X = \mathbb{E}X_n = p, \qquad R_X = \mathbb{E}X_n X_m = p^2, \qquad C_X = \mathbb{E}(X_n - p)(X_m - p) = 0.$$

3 Number of Successes

For the above experiment, let N_n denote the number of successes in first n trials. Then, we have

$$N_n(\omega) = \sum_{i=1}^{n} 1\{\omega_i = S\} = \sum_{i=1}^{n} X_i(\omega).$$

The discrete process $\{N_n(\omega) : n \in \mathbb{N}\}$ is a stochastic process that takes discrete values in $\mathbb{N}_0$. In particular, $N_n \in \{0, \dots, n\}$, i.e. the set of all outcomes is index dependent. Further, $N_n \geq 0$ for all n and is a non-decreasing process, since $N_n = N_{n-1} + 1\{\omega_n = S\}$.

Example 3.1. Example of discrete counting processes.

i_ For products manufactured in an assembly line, N_n indicates the number of defective products in the first n manufactured.

ii_ At a fork on the road, N_n indicates the number of vehicles that turned left for first n vehicles that arrived at the fork.

We can characterize the moments of this stochastic process

$$m_N(n) = \mathbb{E}X_n = np, \qquad \mathrm{Var}\, N_n = \sum_{i=1}^{n} \mathrm{Var}\, X_i = npq.$$

Clearly, this process is not stationary since the first moment is index dependent. In the next lemma, we try to characterize the distribution of random variable N_n.

Theorem 3.2. *The distribution of number of successes N_n in first n trials of a Bernoulli process is given by a Binomial (n, p) distribution*

$$P_n(k) = \binom{n}{k} p^k q^{(n-k)}.$$

Proof. Number of successes N_n is sum of n iid Bernoulli random variables, ahd hence has a Binomial distribution. □

Theorem 3.3. *The stochastic process $(N_n : n \in \mathbb{N})$ has stationary and independent increments.*

Proof. We can look at one increment

$$N_{m+n} - N_m = \sum_{i=1}^{n} X_{m+i}.$$

This increment is a function of $(X_{m+1}, \dots, X_{m+n})$ and hence independent of $(X_1, \dots, X_m)$. The random variable N_m depends solely on $(X_1, \dots, X_m)$ and hence the independence follows. Stationarity follows from the fact that the Bernoulli process X is iid and $N_{m+n} - N_m$ is sum of n iid Bernoulli random variables, and hence has a Binomial (n, p) distribution identical to that of N_n. □

Corollary 3.4. *Let $p \in \mathbb{N}$ and for each $i \in [p]$ let $n_i \in \mathbb{N}, k_i \in \mathbb{N}_0$. For a finite ordered set $S = (n_1, n_1 + n_2, \dots, n_1 + n_2 + \dots + n_p) \subset \mathbb{N}$ and $k_S = (k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_p)$, we have the joint mass function*

$$P_S(k_S) = P(\cap_{i \in [k]} \{N_{n_1 + \dots + n_i} = k_1 + \dots + k_i\}) = \prod_{i=1}^{p} P_{n_i}(k_i).$$

Proof. The result follows from stationary and independent increment property of the counting process N_n. □

Lemma 3.5. *The stochastic process $(N_n : n \in \mathbb{N})$ is homogeneously Markov.*

Proof. Since the process has stationary and independent increments, we have

$$P\{N_{n+m} = k | N_1 = k_1, N_2 = k_2, \dots, N_n = k_n\} = P\{N_{n+m} - N_n = k - k_n\} = P\{N_{n+m} = k | N_n = k_n\}.$$

□

4 Random Walk

Let $X = (X_n \in \mathbb{R}^d : n \in \mathbb{N})$ be an iid random sequence. Let $S_0 = 0$ and $S_n \triangleq \sum_{i=1}^n X_i$, then the process $S = (S_n : n \in \mathbb{N})$ is called a **random walk**. We can think of S_n as the random location of a particle after n steps, where the particle starts from origin and takes steps of size X_i at the ith step.

From previous section, we know following properties of random walks.

Theorem 4.1. *For a random walk $(S_n : n \in \mathbb{N})$ with iid step-size sequence X, the following are true.*

i_ The first two moments are $\mathbb{E}S_n = n\mathbb{E}X_i$ and $\mathrm{Var}[S_n] = n\,\mathrm{Var}[X_i]$.

ii_ Random walk is non-stationary with stationary and independent increments.

iii_ Random walk is homogeneous Markov sequence.

When X is a Bernoulli sequence, with $P\{X_i = 1\} = p = 1 - P\{X_i = -1\}$, the one dimensional random walk S is an integer valued random sequence with unit step-size.

Theorem 4.2. *For a one-dimensional integer valued random walk $(S_n : n \in \mathbb{N})$ with iid unit step size sequence $(X_n : n \in \mathbb{N})$ such that $P\{X_1 = 1\} = p$, the following are true.*

i_ Number of positive steps after n steps is Binomial (n, p).

ii_ $P\{S_n = k\} = \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2}$ for $n+k$ even, and 0 otherwise.

5 Stopping Times

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration on this probability space for an ordered index set T. A random variable $\tau \in \mathcal{F}$ is called a **stopping time** with respect to this filtration if the event $\{\tau \leq t\} \in \mathcal{F}_t$.

Let $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$ for a random process $X = (X_t : t \in T)$. We can consider the ordered index set T as a time sequence. Intuitively, if we observe the process X sequentially, then the event $\{\tau \leqslant t\}$ can be completely determined by the observation $(X_s, s \leqslant t)$ till time t. The intuition behind a stopping time is that it's realization is determined by the past and present events but not by future events.

Example 5.1. Examples of stopping times.

1. For instance, while traveling on the bus, the random variable measuring "time until bus crosses next stop after Majestic" is a stopping time as it's value is determined by events before it happens. On the other hand "time until bus crosses the stop before Majestic" would not be a stopping time in the same context. This is because we have to cross this stop, reach Majestic and then realize we have crossed that point.

2. Let $(N_n : n \in \mathbb{N})$ be the number of successes for an iid Bernoulli process X, then $T_k \triangleq \min\{n \in \mathbb{N} : N_n = k\}$ is a stopping time.

3. For any measurable set $A \in \mathcal{F}$, the hitting time $\min\{n \in N : S_n \in A\}$ of the set A by random walk S is a stopping time adapted to the natural filtration $\mathcal{F}_\bullet = (\mathcal{F}_n = \sigma(X_i, i \leqslant n) : n \in \mathbb{N})$.

For the special case when $T = \mathbb{N}$ is a countable ordered index set, then stopping time can be defined as a random variable N taking countably many values in $\mathbb{N} \cup \{\infty\}$ if for each $n \in \mathbb{N}$, we have the event $\{N = n\} \in \mathcal{F}_n$.

5.1 Properties of stopping time

Lemma 5.2. *Let τ_1, τ_2 be two stopping times with respect to filtration $(\mathcal{F}_t : t \in T)$. Then the following hold true.*

i_ $\min\{\tau_1, \tau_2\}$ is a stopping time.

ii_ If T is separable, then $\tau_1 + \tau_2$ is a stopping time.

Proof. Let $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration, and τ_1, τ_2 associated stopping times.

i. Result follows since the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$.

ii. It suffices to show that the event $\{\tau_1 + \tau_2 \leq t\} \in \mathcal{F}_t$ for $T = \mathbb{R}_+$. To this end, we observe that

$$\{\tau_1 + \tau_2 \leqslant t\} = \bigcup_{s \in \mathbb{Q}_+ :\, s \leqslant t} \{\tau_1 \leqslant t - s, \tau_2 \leqslant s\} \in \mathcal{F}_t.$$

$\square$