Lecture-20: Stopping Time σ-algebra

1 Wald's Lemma

Lemma 1.1 (Wald's Lemma). *Consider a random walk $(S_n : n \in \mathbb{N})$ with iid step-sizes $(X_n : n \in \mathbb{N})$ having finite $\mathbb{E}|X_1|$, Let N be a finite mean stopping time adapted to the natural filtration $\mathcal{F}_\bullet = (\mathcal{F}_n = \sigma(X_1, \dots, X_n) : n \in \mathbb{N})$ Then,*

$$\mathbb{E}S_N = \mathbb{E}X_1\mathbb{E}N.$$

Proof. From the independence of step sizes, it follows that X_n is independent of $\mathcal{F}_{n-1}$. Next we observe that $\{N \geqslant n\} = \{N > n-1\} \in \mathcal{F}_{n-1}$, and hence $\mathbb{E}[X_n 1_{\{N\geqslant n\}}] = \mathbb{E}X_n\mathbb{E}1_{\{N\geqslant n\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{N} X_n = \mathbb{E}\sum_{n\in\mathbb{N}} X_n 1_{\{N\geqslant n\}} = \sum_{n\in\mathbb{N}} \mathbb{E}X_n\mathbb{E}\left[1_{\{N\geqslant n\}}\right] = \mathbb{E}X_1\mathbb{E}\left[\sum_{n\in\mathbb{N}} 1_{\{N\geqslant n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[N].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it since the summand is positive and we apply monotone convergence theorem. □

Corollary 1.2. *Consider the stopping time $T_i = \min\{n \in \mathbb{N} : S_n = i\}$ for an integer random walk S with iid steps X. Then, the mean of stopping time $\mathbb{E}T_i = i/\mathbb{E}X_1$.*

A Wald type result for a random sum $S_N = \sum_{n=1}^{N} X_n$ of iid random variables $X = (X_n : n \in \mathbb{N})$, when N is independent of the sequence X is trivial to obtain, since

$$\mathbb{E}[S_N] = \mathbb{E}[\mathbb{E}[S_N|N]] = \mathbb{E}[N\mathbb{E}X_1] = \mathbb{E}N\mathbb{E}X_1.$$

When the random variable N is not independent of the underlying process X, the linearity of expectation of the random sum S_N does not always hold. For example, let's take our counting process $(N_n : n \in \mathbb{N})$ for the number of successes in an iid Bernoulli trial. We take the discrete random time $\tau' = K \wedge \max\{n \in \mathbb{N} : N_n = 1\}$. Then, $\mathbb{E}N_{\tau'} = 1$, however $P\{\tau' = K\} = 1$ and hence $\mathbb{E}\tau'\mathbb{E}X_1 = Kp \neq 1$ for all $p \neq 1/K$. However, when the random variable N is a stopping time with respect to the natural filtration for this process, then even though N is not independent of the sequence X, the linearity holds. For the same counting process, we can take the stopping time $\tau = \min\{N_n = 1\}$. Then,

$$1_{\{\tau=i\}}P(\{X_{[i]} = (0,0,\dots,1)\}|\sigma(\tau)) = P(\{X_{[i]} = (0,0,\dots,1)\}|\tau = i) = 1 \neq (q)^{i-1}p = \prod_{j=1}^{i-1} P\{X_j = 0\}P\{X_i = 1\}.$$

Time for first success is a geometrically distributed random variable with mean $1/\mathbb{E}X_1$, hence we can check that $\mathbb{E}N_\tau = 1 = \mathbb{E}X_1\mathbb{E}\tau$.

2 Stopping time σ-algebra

We wish to define a σ-algebra consisting information of the process till a random time τ. For a countable stopping time τ, what we want is something like $\sigma(X_1, \dots, X_\tau)$. But that doesn't make sense, since the random time is a random variable itself. When τ is a stopping time, the event $\{\tau \leqslant t\} \in \mathcal{F}_t$. What makes sense is the set of all measurable sets whose intersection with $\{\tau \leqslant t\}$ belongs to $\mathcal{F}_t$ for all $t \geqslant 0$.

For a stopping time $\tau : \Omega \to \mathbb{R}_+$ adapted to the filtration $\mathcal{F}_\bullet$, the **stopping time σ-algebra** is defined as

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t, \forall t \geqslant 0\}.$$

One can check that $\mathcal{F}_\tau$ is indeed a σ-algebra. Further, $\mathcal{F}_\tau$ has information up to the random time τ. That is, it is a collection of measurable sets that are determined by the process till time τ. Any measurable set $A \in \mathcal{F}$ can be written as $A = (A \cap \{\tau \leqslant t\}) \cup (A \cap \{\tau > t\})$. All such sets A such that $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ is a member of the stopped σ-algebra.

Lemma 2.1. *Let $\mathcal{F}_\bullet$ be the natural filtration associated with the process $(X_t : t \in T)$, and τ be the associated stopping time. Let $Y_t = X_{\tau \wedge t}$, that is $Y_s = X_s 1_{\{s \leqslant \tau\}} + X_\tau 1_{\{s > \tau\}}$. Then $\mathcal{F}_\tau = \sigma(Y_s, s \leqslant t)$.*

Proof. $\square$

Lemma 2.2. *Let τ, τ_1, τ_2 be stopping times adapted to a filtration $\mathcal{F}_\bullet$. Then, the following are true.*

i₋ $\sigma(\tau) \subseteq \mathcal{F}_\tau$.

ii₋ If $\tau_1 \leqslant \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

Proof. Let τ be a stopping time adapted to a filtration $\mathcal{F}_\bullet$. Then, for any $t \geqslant 0$, we have $\{\tau \leqslant t\} \in \mathcal{F}_t$

i₋ We show that for any $s \geqslant 0$, the event $\{\tau \leqslant s\} \in \mathcal{F}_\tau$. This is true because for any $t \geqslant 0$

$$\{\tau \leqslant s\} \cap \{\tau \leq t\} = \{\tau \leqslant s \wedge t\} \in \mathcal{F}_t.$$

ii₋ From the hypothesis, we have $\{\tau_2 \leqslant t\} \subseteq \{\tau_1 \leqslant t\}$ almost surely. Let $A \in \mathcal{F}_{\tau_1}$ then $A \cap \{\tau_1 \leqslant t\} \in \mathcal{F}_t$ for all $t \geqslant 0$. Further, we see that $A \cap \{\tau_2 \leqslant t\} = A \cap \{\tau_2 \leqslant t\} \cap \{\tau_1 \leqslant t\} \in \mathcal{F}_t$ for all $t \geqslant 0$.

$\square$

3 Strong Markov property

Let X be a real valued Markov process adapted to a filtration $\mathcal{F}_\bullet$. Let τ be an almost surely finite stopping time with respect to to this filtration, then the process X is called **strongly Markov** if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$P(\{X_{t+\tau} \leqslant x\} | \mathcal{F}_\tau) = P(\{X_{t+\tau} \leqslant x\} | \sigma(X_\tau)).$$

Lemma 3.1. *Let $(X_t : t \in T)$ be any Markov process adapted to filtration $(\mathcal{F}_t : t \in T)$. For any almost surely finite stopping time τ with respect to this filtration that takes only countably many values, Markov process X is strongly Markov at this stopping time τ.*

Proof. Let $I \subseteq T$ be the countable set such that $\{\tau \in I\} = \Omega$. Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau = i\} \in \mathcal{F}_i$ for all $i \in I$. Then,

$$\begin{aligned}\mathbb{E}[1_A 1_{\{X_{t+\tau} \leqslant x\}}] &= \sum_{i \in I} \mathbb{E}[1_{A \cap \{X_{t+\tau} \leqslant x\} \cap \{\tau = i\}}] = \sum_{i \in I} \mathbb{E}[\mathbb{E}[1_{A \cap \{X_{t+i} \leqslant x\} \cap \{\tau = i\}} | \mathcal{F}_i]] = \sum_{i \in I} \mathbb{E}[1_{A \cap \{\tau = i\}} \mathbb{E}[1_{\{X_{t+i} \leqslant x\}} | \mathcal{F}_i]] \\ &= \sum_{i \in I} \mathbb{E}[1_{A \cap \{\tau = i\}} \mathbb{E}[1_{\{X_{t+i} \leqslant x\}} | \sigma(X_i)]] = \mathbb{E}[1_A \sum_{i \in I} 1_{\{\tau = i\}} \mathbb{E}[1_{\{X_{t+i} \leqslant x\}} | \sigma(X_i)]] = \mathbb{E}[1_A \mathbb{E}[1_{\{X_{t+\tau} \leqslant x\}} | \sigma(X_\tau)]].\end{aligned}$$

The result follows since $P(\{X_{t+\tau} \leqslant x\} | \sigma(X_\tau)) \in \mathcal{F}_\tau$. $\square$

Corollary 3.2. *Any Markov process on countable index set T is strongly Markov.*

Proof. For a countable index set T, all associated stopping times assume at most countably many values. $\square$

Corollary 3.3. *Let τ be an almost surely finite stopping time with respect to the natural filtration $\mathcal{F}_\bullet$ of an iid random sequence X. Then $(X_{\tau+1}, \dots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ for each $n \in N$ and identically distributed to $(X_1, \dots, X_n)$.*