Lecture-21: Success Instants in Bernoulli Trials

1 Success Instants Process

For the above experiment, let T_k denote the trial number corresponding to kth success. Then $(T_k(\omega) : k \in \mathbb{N})$ is a discrete random sequence defined inductively as

$$T_1 = \inf\{n \in \mathbb{N} : X_n(\omega) = 1\}, \qquad\qquad T_{k+1}(\omega) = \inf\{n > T_k : X_n(\omega) = 1\}.$$

For example, if $X = (0, 1, 0, 1, 1, \dots)$, then $T_1 = 2, T_2 = 4, T_3 = 5$ and so on. Clearly, $T_k \geq k$, and the discrete process $(T_k(\omega) : k \in \mathbb{N})$ is a stochastic process that takes discrete values in $\{k, k+1, \dots\}$. This implies that the process is not stationary since one can find two time points where the means differ. We next observe the following inverse relationship between number of nth successful trial T_n, and number of successes in n trials.

Lemma 1.1 (Inverse relationship). *The following inverse relationship holds between time of successes and number of successes*

$$\{T_k \leq n\} = \{N_n \geq k\}.$$

Proof. To see the first equality, we observe that $\{T_k \leq n\}$ is the set of outcomes, where $X_{T_1} = X_{T_2} = \cdots = X_{T_k} = 1$, and $\sum_{i=1}^{T_k} X_i = k$. Hence, we can write the number of successes in first n trials as

$$N_n = \sum_{i=1}^{n} X_i = \sum_{i > T_k} X_i + \sum_{i=1}^{T_k} X_i \geq k.$$

Conversely, we notice that we can re-write the number of trials for ith success as

$$T_i = \inf\{m \in \mathbb{N} : N_m = i\}.$$

Since N_n is non-decreasing in n, it follows that for the set of outcome such that $\{N_n \geq k\}$, there exists $m \leq n$ such that $T_k = m \leq n$. $\qquad\square$

Corollary 1.2. *The time of nth success can be written in terms of the counting and the Bernoulli process as*

$$\{T_k = n\} = \{N_{n-1} = k - 1, X_n = 1\}.$$

Proof. For the second equality, we observe that

$$\{T_k = n\} = \{T_k \leq n\} \cap \{T_k \leq n-1\}^c = \{N_n \geq k\} \cap \{N_{n-1} \geq k\}^c = \{N_{n-1} = k - 1, N_n = k\}.$$

$\qquad\square$

Corollary 1.3. *The marginal distribution of process $(T_k : k \in \mathbb{N})$ is*

$$P\{T_k \leq n\} = \sum_{j=k}^{\infty} \binom{n}{j} p^j q^{(n-j)}, \qquad\qquad P\{T_k = n\} = \binom{n-1}{k-1} p^k q^{n-k}.$$

Proof. Using the inverse relationship, we can write the marginal distribution of process $(T_k : k \in \mathbb{N})$ in terms of the marginal of the process $(N_n : n \in \mathbb{N})$ as

$$P\{T_k \leq n\} = P\{N_n \geq k\} = \sum_{j \geq k} P_n(j), \qquad P\{T_k = n\} = P\{X_n = 1\} P\{N_{n-1} = k - 1\} = p\dot{P}_{n-1}(k-1).$$

$\qquad\square$

As we had observed before this process is not stationary, because we can now explicitly compute the index dependent first moment of the process. Further, we realize that it is not straightforward to characterize moments of T_k from its marginal distribution.

1.1 Properties of success instants process

Let $\mathcal{F}_\bullet$ be the natural filtration associated with the Bernoulli process X. It follows from the inverse relation that $\{T_k = m\} \in \mathcal{F}_m$, and hence T_k is a stopping time with respect to the filtration $\mathcal{F}_\bullet$ for each $k \in \mathbb{N}$. For each stopping time T_k, we can associate a stopping time σ-algebra $\mathcal{F}_{T_k}$. Since $(T_k : k \in \mathbb{N})$ is a non-decreasing random sequence, the collection $(\mathcal{F}_{T_k} : k \in \mathbb{N})$ is a filtration where $T_k \in \mathcal{F}_{T_k}$.

Lemma 1.4. *The time of success process $(T_k : k \in \mathbb{N})$ has stationary and independent increments.*

Proof. It suffices to show that the kth increment $T_{k+1} - T_k$ is independent of T_{k-1} and the distribution of the kth increment is stationary. The stopping time T_k has countable outcomes, and is finite almost surely by induction over k. Since $T_k - T_{k-1} \in \sigma(X_{T_{k-1}+i}, i \in \mathbb{N})$, it is independent of $\mathcal{F}_{T_{k-1}}$. Further, for each $n \in \mathbb{N}$

$$\{T_k - T_{k-1} = n\} = \{X_{T_{k-1}+1} = \cdots = X_{T_{k-1}+n-1} = 0, X_{T_{k-1}+n} = 1\}.$$

Since the joint distribution of $(X_{T_{k-1}+1}, \ldots, X_{T_{k-1}+n})$ is identical to the joint distribution of $(X_1, \ldots, X_n)$, we have

$$P\{T_k - T_{k-1} = n\} = pq^{n-1}, \ n \in \mathbb{N}.$$

$\square$

Corollary 1.5. *We can write the first two moments of T_k as*

$$\mathbb{E}T_k = \frac{k}{p}, \qquad\qquad \mathrm{Var}[T_k] = \frac{kq}{p^2}.$$

Proof. Since, the increment in time of success follows a geometric distribution with success probability p, we have mean of the increment $\mathbb{E}(T_i - T_{i-1}) = 1/p$, and the variance $\mathrm{Var}[T_i - T_{i-1}] = q/p^2$ for each $i \in \mathbb{N}$. Result follows from the linearity of mean, linearity of variance for independent random variables, and the telescopic sum $T_k = \sum_{i=1}^{k}(T_i - T_{i-1})$. $\square$

This shows that stationary and independent increments is a powerful property of a process and makes the process characterization much simpler. Next, we show an additional property of this process.

Lemma 1.6. *The increments of time of success process $(T_k : k \in \mathbb{N})$ are memoryless.*

Proof. It follows from the property of a geometric distribution, that for positive integers m, n

$$P(\{T_{k+1} - T_k > m+n\}|\{T_{k+1} - T_k > m\}) = \frac{P\{T_{k+1} - T_k > m+n\}}{P\{T_{k+1} - T_k > m\}} = q^n = P\{T_{k+1} - T_k > n\}.$$

$\square$

Example 1.7. Examples of renewal process.

  i₋ For products manufactured in an assembly line, T_k indicates the number of products inspected for kth defective product to be detected.

  ii₋ At a fork on the road, T_k indicates the number of vehicles that have arrived at fork for kth left turning vehicle.

Lemma 1.8. *The time of success process $(T_k : k \in \mathbb{N})$ is Markov.*

Proof. Consider the natural filtration $\mathcal{G}_\bullet = (\mathcal{G}_k = \sigma(T_1, \ldots, T_k) \subseteq \mathcal{F}_{T_k} : k \in \mathbb{N})$ generated by the process $(T_k : k \in \mathbb{N})$. We can write the event $\{T_k = n\}$ as a disjoint union $\cup_{m<n}\{T_k - T_{k-1} = n-m, T_{k-1} = m\}$. In addition, the increment $\{T_k - T_{k-1} = n - m\}$ is independent of $\mathcal{F}_{T_k}$ and hence of $\mathcal{G}_k$, and $T_{k-1} \in \mathcal{G}_k$. Therefore, we have

$$\mathbb{E}[1_{\{T_k=n\}}|\mathcal{G}_{k-1}] = \sum_{m=1}^{n-1}\mathbb{E}[1_{\{T_k-T_{k-1}=n-m\}}1_{\{T_{k-1}=m\}}|\mathcal{G}_{k-1}] = \sum_{m=1}^{n-1}1_{\{T_{k-1}=m\}}\mathbb{E}[1_{\{T_k-T_{k-1}=n-m\}}] = \mathbb{E}[1_{\{T_k=n\}}|\sigma(T_{k-1})].$$

$\square$