Lecture-23: Discrete Time Markov Chains

1 Introduction

We have seen that *iid* sequences are easiest discrete time random processes. However, they don't capture correlation well. We saw some example of Markov processes where $X_n = X_{n-1} + Z_n$, and $(Z_n : n \in \mathbb{N})$ is an iid sequence, independent of the initial state X_0. We can generalize this to arbitrary functions. Hence, we look at the discrete time stochastic processes of the form

$$X_{n+1} = f(X_n, Z_{n+1}).$$

For a countable set S, a stochastic process $(X_n \in S : n \in \mathbb{N}_0)$ is called a **discrete time Markov chain (DTMC)** if for all positive integers $n \in \mathbb{N}_0$ and all states $j \in S$, the process X satisfies the Markov property

$$P(\{X_{n+1} = j\}|\mathcal{F}_n) = P(\{X_{n+1} = j\}|\sigma(X_n)),$$

where $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ is the natural filtration. Since the state space S is countable, the probability $P(\{X_{n+1} = j\}|\sigma(X_n))$ can be written as

$$P(\{X_{n+1} = j\}|\sigma(X_n)) = \sum_{i \in S} 1_{\{X_n = i\}} P(\{X_{n+1} = j\}|\sigma(X_n)) = \sum_{i \in S} 1_{\{X_n = i\}} P(\{X_{n+1} = j\}|\{X_n = i\}).$$

That is the probability of a discrete time Markov chain X being in state j at time $n+1$ from a state i at time n, is determined by the **transition probability** denoted by

$$p_{ij}(n) \triangleq P(\{X_{n+1} = j\}|\{X_n = i\}).$$

1.1 Homogeneous Markov chain

In general, not much can be said about Markov chains with index dependent transition probabilities. Hence, we consider the simpler case where the transition probabilities $p_{ij}(n) = p_{ij}$ are independent of the index. We call such DTMC as **homogeneous** and call the linear operator $P = (p_{ij} : i, j \in \mathbb{E})$ the **transition matrix**. The transition matrix P is stochastic matrix.

For all states $i, j \in S$, if a non-negative matrix $A \in \mathbb{R}_+^{E \times E}$ has the following property

$$a_{ij} \geq 0, \qquad \sum_{j \in S} a_{ij} \leq 1,$$

then it is called a **sub-stochastic** matrix. If the second property holds with equality, then it is called a **stochastic** matrix. If in addition, A^T is stochastic, then A is called **doubly stochastic**.

1.2 Transition graph

A transition matrix P is sometimes represented by a directed graph $G = (E, \{[i,j\rangle \in S \times E : p_{ij} > 0\})$, where the state space E is the set of nodes and $[i,j\rangle$. In addition, this graph has a weight p_{ij} on each edge $e = [i,j\rangle$.

2 Chapman Kolmogorov equations

We can define n-step transition probabilities for $i, j \in S$ and $m, n \in \mathbb{N}$

$$p_{ij}^{(n)} \triangleq P(\{X_{n+m} = j\}|\{X_m = i\}).$$

It follows from the Markov property and law of total probability that

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}.$$

We can write this result compactly in terms of transition probability matrix P as $P^{(n)} = P^n$. Let $\nu \in \mathbb{R}_+^E$ is a probability vector such that

$$\nu_n(i) = P\{X_n = i\}.$$

Then, we can write this vector ν_n in terms of initial probability vector ν_0 and the transition matrix P as

$$\nu_n = \nu_0 P^n.$$

2.1 Strong Markov property (SMP)

Let T be an integer valued stopping time with respect to the stochastic process X such that $P\{T < \infty\} = 1$. Then for all $i_0, \dots, i_{n-1}, \dots, i, j \in S$, the process X satisfies the **strong Markov property** if

$$P(\{X_{T+1} = j\} | \{X_T = i, \dots, X_0 = i_0\}) = P(\{X_{T+1} = j\} | \{X_T = i\}).$$

Lemma 2.1. *Markov chains satisfy the strong Markov property.*

Proof. Let X be a Markov chain and $A = \{X_T = i, \dots, X_0 = i_0\}$. Then, we have

$$P(\{X_{T+1} = j\}|A) = \frac{\sum_{n \in \mathbb{N}_0} P(\{X_{T+1} = j, A, T = n\})}{P(A)} = \sum_{n \in \mathbb{N}_0} p_{ij} \frac{P(A, T = n)}{P(A)} = p_{ij}.$$

This equality follows from the fact that $\{T = n\}$ is completely determined by $\{X_0, \dots, X_n\}$ □

As an exercise, if we try to use the Markov property on arbitrary random variable T, the SMP may not hold. For example, define a non-stopping time T for $j \in S$

$$T = \inf\{n \in \mathbb{N}_0 : X_{n+1} = j\}.$$

In this case, we have

$$P\{X_{T+1} = j | X_T = i, \dots, X_0 = i_0\} = 1\{p_{ij} > 0\} \neq P\{X_1 = j | X_0 = i\} = p_{ij}.$$

A useful application of the strong Markov property is as follows. Let $i_0 \in S$ be a fixed state and $\tau_0 = 0$ Let τ_n denote the stopping times at which the Markov chain visits i_0 for the nth time. That is,

$$\tau_n = \inf\{n > \tau_{n-1} : X_n = i_0\}.$$

Then $\{X_{\tau_n + m} : m \in \mathbb{N}_0\}$ is a stochastic replica of $\{X_m : m \in \mathbb{N}_0\}$ with $X_0 = i_0$ and can be studied as a regenerative process.