Lecture-24: Markov Chains: Hitting and Recurrence Times

1 Hitting and Recurrence Times

Let X be a time-homogeneous Markov chain on state space S with transition probability matrix P. For each $j \in S$, we can define the first hitting time to state j after $n = 0$, as

$$H_j \triangleq \inf\{n \in \mathbb{N} : X_n = j\}.$$

For each $n \in \mathbb{N}$, we can write the probability of first visit to state j at time n from the initial state i, as

$$f_{ij}^{(n)} \triangleq P(H_j = n | X_0 = i).$$

The probability that the Markov chain X hits state j eventually, starting from initial state i is

$$f_{ij} \triangleq P(H_j < \infty | X_0 = i) = P(\cup_{n\in\mathbb{N}}\{H_j = n\}|X_0 = i) = \sum_{n\in\mathbb{N}} P(H_j = n|X_0 = i) = \sum_{n\in\mathbb{N}} f_{ij}^{(n)}.$$

The distribution $((f_{ij}^{(n)} : n \in \mathbb{N}), 1 - f_{ij})$ is called the **first passage time distribution** for hitting state j from initial state i. The distribution $((f_{ii}^{(n)} : n \in \mathbb{N}), 1 - f_{ii})$ is called the **first recurrence time distribution** for return to initial state i. A state is called **recurrent** if $f_{ii} = 1$, and is called **transient** if $f_{ii} < 1$. For a recurrent state $i \in S$, we can defined **mean recurrence time** as

$$\mu_{ii} \triangleq \sum_{n\in\mathbb{N}} n f_{ii}^{(n)}.$$

If the mean recurrence time for a recurrent state i is finite then the state i is called positive recurrent, and **null recurrent** otherwise. We would donate the conditional probability and conditional expectation of a measurable event A starting from state i as

$$P_i(A) \triangleq P(A|\{X_0 = i\}), \qquad \mathbb{E}_i 1_A \triangleq \mathbb{E}[1_A|\{X_0 = i\}].$$

Proposition 1.1. *The total number of visits to a state $j \in S$ after starting from initial state i is denoted by $N_j = \sum_{n\in\mathbb{N}} 1\{X_n = j\}$. Then, for each $m \in \mathbb{N}_0$, we have*

$$P_i\{N_j = m\} = \begin{cases} 1 - f_{ij}, & m = 0,\\ f_{ij} f_{jj}^{m-1}(1 - f_{jj}), & m \in \mathbb{N}. \end{cases}$$

Proof. Conditioned on $X_0 = i$, the first passage time H_j to state j being finite is a Bernoulli random variable with probability f_{ij}. The time of the mth return to the state j is a recurrence time for each $m \in \mathbb{N}_0$. From strong Markov property, each return to state j is independent of the past. Hence, each return to state j in a finite time is an *iid* Bernoulli random variable with probability f_{jj}. It follows that the number of recurrences to state j is the time for first failure to return. Conditioned on initial state being $X_0 = j$, the distribution of N_j is geometric random variable with success probability $1 - f_{jj}$. □

Proof. Another way to see this, is to consider $P_i(N_j > m)$. Let $H_j^{(k)}$ be the kth hitting time of state j, then

$$P_i(N_j > m) = \sum_{n_1,n_2,\dots,n_m\in\mathbb{N}} P_i(H_j^{(1)} = n_1)P_j(H_j^{(2)} = n_2)\dots P_j(H_j^{(m)} = n_m) = f_{ij} f_{jj}^{m-1}.$$

□

Corollary 1.2. *The mean number of visits to state j, starting from a state i is*

$$\mathbb{E}_i N_j = \begin{cases} \frac{f_{ij}}{1-f_{jj}}, & f_{jj} < 1,\\ \infty, & f_{jj} = 1. \end{cases}$$

Corollary 1.3. *For a Markov chain X, $P_i\{N_j < \infty\} = 1\{f_{jj} < 1\}$.*

Proof. We can write the event $\{N_j < \infty\}$ as disjoint union of events $\{N_j = n\}$, to get

$$P_i\{N_j \in \mathbb{N}_0\} = \sum_{n\in\mathbb{N}_0} P_i\{N_j = n\} = 1\{f_{jj} < 1\}.$$

$\square$

Remark 1. In particular, this corollary implies the following consequences.

i_ A transient state is visited a finite amount of times almost surely.

ii_ A recurrent state is visited infinitely often almost surely.

iii_ In a finite state Markov chain, not all states may be transient.

Proposition 1.4. *A state j is recurrent iff $\sum_{k\in\mathbb{N}} p_{jj}^{(k)} = \infty$, and transient iff $\sum_{k\in\mathbb{N}} p_{jj}^{(k)} < \infty$.*

Proof. For any state $j \in S$, we can write $p_{ii}^{(k)} = P_i\{X_k = i\} = \mathbb{E}_i 1\{X_k = i\}$. Using monotone convergence theorem to exchange expectation and summation, we obtain

$$\sum_{k\in\mathbb{N}} p_{ii}^{(k)} = \mathbb{E}_i \sum_{k\in\mathbb{N}} 1\{X_k = i\} = \mathbb{E}_i N_i.$$

Thus, $\sum_{k\in\mathbb{N}} p_{ii}^{(k)}$ represents the expected number of returns $\mathbb{E}_i N_i$ to a state i starting from state i, which we know to be finite if the state is transient and infinite if the state is recurrent. $\square$

Corollary 1.5. *For a transient state $j \in S$, the following limits hold $\lim_{n\in\mathbb{N}} p_{ij}^{(n)} = 0$, and $\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{ij}^{(k)}}{n} = 0$.*

Proof. For a transient state $j \in S$ and any state $i \in S$, we have $\mathbb{E}_i N_j = \sum_{n\in\mathbb{N}} p_{ij}^{(n)} < \infty$. $\square$

Theorem 1.6. *Let $i, j \in S$ be such that $f_{ij} = 1$ and j is recurrent. Then, $\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{ij}^{(k)}}{n} = \frac{1}{\mu_{jj}}$.*

Proof. Let $N_j(n) = \sum_{k=1}^n 1\{X_k = j\}$ be the number of visits to state j in n steps of the Markov process X. Hence, we have $\sum_{\ell=1}^{N_j(n)+1} H_j^{(\ell)} > n$. By Wald's Lemma, we have $\mathbb{E}_j(N_j(n)+1)\mu_{jj} > n$. Taking limits, we obtain $\liminf_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{jj}^{(k)}}{n} \geqslant \frac{1}{\mu_{jj}}$.

For the converse, we can use a counting process with truncated recurrence times $\bar{H}_j^\ell = M \wedge H_j^\ell$. It follows that $\bar{N}_j(n) \geqslant N_j(n)$ sample path wise, and $\bar{\mu}_{jj} \triangleq \mathbb{E}_j \bar{H}_j \leqslant \mathbb{E}_j H_j = \mu_{jj}$. Further, we have $\sum_{\ell=1}^{\bar{N}_j(n)+1} \bar{H}_j \leqslant n + M$. From Wald's Lemma, we have

$$\mathbb{E}_j(N_j(n)+1)\bar{\mu}_{jj} \leqslant \mathbb{E}_j(\bar{N}_j(n)+1)\bar{\mu}_{jj} \leqslant n + M.$$

Taking limits, we obtain $\limsup_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{ij}^{(k)}}{n} \leqslant \frac{1}{\bar{\mu}_{jj}}$. Letting M grow arbitrarily large, we obtain the upper bound. Further, we observe that $p_{ij}^{(k)} = \sum_{s=0}^{k-1} f_{ij}^{(k-s)} p_{jj}^{(s)}$. Since $1 = f_{ij} = \sum_{k\in\mathbb{N}} f_{ij}^{(k)}$, we have

$$\sum_{k=1}^n p_{ij}^{(k)} = \sum_{k=1}^n \sum_{s=0}^{k-1} f_{ij}^{(k-s)} p_{jj}^{(s)} = \sum_{s=0}^{n-1} p_{jj}^{(s)} \sum_{k-s=1}^{n-s} f_{ij}^{(k-s)} = \sum_{s=0}^{n-1} p_{jj}^{(s)} - \sum_{s=0}^{n-1} p_{jj}^{(s)} \sum_{k>n-s} f_{ij}^{(k)}.$$

Since the series $\sum_{k\in\mathbb{N}} f_{ij}^{(k)}$ converges, we get

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{ij}^{(k)}}{n} = \lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^n p_{jj}^{(k)}}{n}.$$

$\square$