Lecture-25: Markov Chains: Class Properties

1 Communicating classes

State $j \in S$ is said to be **accessible** from state $i \in S$ if $p_{ij}^{(n)} > 0$ for some $n \in \mathbb{N}_0$, and denoted by $i \to j$. If two states $i, j \in S$ are accessible to each other, they are said to **communicate** with each other, denoted by $i \leftrightarrow j$. A set of states that communicate are called a **communicating class**. A communicating class $\mathcal{C}$ is called **closed** if no edges leave this class. That is, for all $i \in \mathcal{C}$ and $j \in \mathcal{C}$, we have $p_{ij} = 0$. An **open** communicating class is not closed.

Proposition 1.1. *Communication is an equivalence relation.*

Proof. Relation on state space S is a subset of product of sets $S \times S$. Communication is a relation on state space S, as it relates two states $i, j \in S$. Reflexivity and symmetry are obvious for this relation. For transitivity, suppose $i \leftrightarrow j$ and $j \leftrightarrow k$. Let $m, n \in \mathbb{N}_0$ such that $p_{ij}^{(m)} > 0$ and $p_{jk}^{(n)} > 0$. Then by Chapman Kolmogorov, we have

$$p_{ik}^{(m+n)} = \sum_{l \in S} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0.$$

This implies $i \to k$, and using similar arguments one can show that $k \to i$, and the transitivity follows. ☐

Hence the communication relation partitions state space S into equivalence classes. Each equivalence class is called a **communicating class**. A property of states is said to be a **class property** if for each communicating class $\mathcal{C}$, either all states in $\mathcal{C}$ have the property, or none do. A Markov chain with a single class is called an **irreducible** Markov chain.

1.1 Periodicity

Let $A_i = \{n \in \mathbb{N} : p_{ii}^{(n)} > 0\}$ for any state $i \in S$. The set A_i is closed under addition, that is if $m, n \in A_i$, then $m + n \in A_i$. The **period** of state i is defined as

$$d(i) = \gcd(A_i).$$

We define $d(i) = \infty$, if $p_{ii}^{(n)} = 0$ for all $n \in \mathbb{N}$. A state $i \in S$ is called **aperiodic** if the period $d(i)$ is 1.

Proposition 1.2. *If $i \leftrightarrow j$, then $d(i) = d(j)$. That is, periodicity is a class property.*

Proof. Let m and n be such that $p_{ij}^{(m)} p_{ji}^{(n)} > 0$. Suppose $p_{ii}^{(s)} > 0$. Then

$$p_{jj}^{(n+m)} \geq p_{ji}^{(n)} p_{ij}^{(m)} > 0, \qquad p_{jj}^{(n+s+m)} \geq p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} > 0.$$

Hence $d(j) | n + m$ and $d(j) | n + s + m$, and hence $d(j) | s$. It follows that $d(j) | d(i)$. By symmetrical arguments, we get $d(i) | d(j)$. Hence $d(i) = d(j)$. ☐

An irreducible Markov chain is called **aperiodic** if the single communicating class is aperiodic.

Lemma 1.3. *For an aperiodic Markov chain, for each state $i \in S$, there exists $n \in \mathbb{N}$ such that $p_{ii}^{(m)} > 0$ for all $m \geqslant n$.*

Proof. For each $i \in S$, consider $A_i = \{n \in \mathbb{N} : p_{ii}^{(n)} > 0\}$. This set is closer under addition, that is if $m, n \in A_i$, then $m + n \in A_i$. From aperiodicity of the Markov chain, $\gcd(A_i) = 1$, and the result follows. ☐

Corollary 1.4. *For each pair of states $i, j \in S$ of an irreducible and aperiodic Markov chain, there exists $n \in \mathbb{N}$ such that $p_{ij}^{(m)} > 0$ for all $m \geqslant n$.*

Proof. From irreducibility, it suffices to show that for all states $i \in S$, there exists $n \in \mathbb{N}$, such that $p_{ii}^{(m)} > 0$ for all $m \geqslant n$. ☐

1.2 Transient and recurrent states

Proposition 1.5. *Transience and recurrence are class properties.*

Proof. Let us start with proving recurrence is a class property. Let i be a recurrent state and let $i \leftrightarrow j$. Hence there exist some $m, n > 0$, such that $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$. As a consequence of the recurrence, $\sum_{s\in\mathbb{N}} p_{ii}^{(s)} = \infty$. It follows that j is recurrent by observing

$$\sum_{s\in\mathbb{N}} p_{jj}^{(m+n+s)} \geq \sum_{s\in\mathbb{N}} p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} = \infty.$$

Now, if i were transient instead, we conclude that j is also transient by the following observation

$$\sum_{s\in\mathbb{N}} p_{jj}^{(s)} \leq \frac{\sum_{s\in\mathbb{N}} p_{ii}^{(m+n+s)}}{p_{ji}^{(n)} p_{ij}^{(m)}} < \infty.$$

□

Lemma 1.6. *If j is recurrent, then for any state i such that $j \to i$, we have $i \to j$ and $f_{ij} = 1$.*

Proof. Since $j \to i$, there exists an $n \in \mathbb{N}$ such that $p_{ji}^{(n)} > 0$. Hence, there exists an $m \leqslant n$, such that $f_{ji}^{(m)} > 0$. Suppose $f_{ij} < 1$, then $1 - f_{jj} \geqslant f_{ji}^{(m)}(1 - f_{ij}) > 0$. This implies j is transient, which is a contradiction. It follows that $i \to j$. □

Corollary 1.7. *Let $i, j \in S$ be in the same communicating class and j is recurrent. Then,* $\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{ij}^{(k)}}{n} = \frac{1}{\mu_{jj}}$. *Furthermore, if j is aperiodic, then* $\lim_{n\in\mathbb{N}} p_{ij}^{(n)} = \frac{1}{\mu_{jj}}$.

Theorem 1.8. *The states in a communicating class are of one of the following types; all transient, or all null recurrent, or all positive recurrent.*

Proof. It suffices to show that if i, j belong to the same communicating class and j is null recurrent, then i is null recurrent as well. We take $r, s \in \mathbb{N}$, such that $p_{ji}^{(r)} p_{ij}^{(s)} > 0$. It follows that $p_{jj}^{r+\ell+s} \geqslant p_{ji}^{(r)} p_{ii}^{(\ell)} p_{ij}^{(s)}$ for all $\ell \in \mathbb{N}$. Hence, for any $n > r + s$, we have

$$\frac{1}{n}\sum_{k=1}^{n} p_{jj}^{(k)} \geqslant \frac{1}{n}\sum_{k=r+s+1}^{n} p_{jj}^{(k)} \geqslant \left(\frac{n-r-s}{n}\right)\left(\frac{1}{n-r-s}\sum_{\ell=1}^{n-r-s} p_{ii}^{(\ell)}\right) p_{ji}^{(r)} p_{ij}^{(s)}.$$

Since j is null recurrent LHS goes to zero as n increases, which implies $\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{\ell=1}^{n} p_{ii}^{(\ell)} = 0$. Hence, i is null recurrent as well. □

Theorem 1.9. *Open communicating classes are transient.*

Proof. If $\mathcal{C}$ is an open communicating class, then there exists $i \in \mathcal{C}$ and $j \notin \mathcal{C}$ such that $p_{ij} > 0$. Since $\mathcal{C}$ is a communicating class, and $i \in \mathcal{C}, j \notin \mathcal{C}$, we have $p_{ji}^{(n)} = 0$ for all $n \in \mathbb{N}$. Hence, $f_{ji} = 0$. Further, we have

$$\begin{aligned} f_{ii} = P_i(H_i < \infty) &= P_i(H_i = 1) + \sum_{\ell\in S\setminus\{i\}} P_i(H_i \geqslant 2, X_1 = \ell) = p_{ii} + p_{ij}P_j(H_i < \infty) + \sum_{\ell\in S\setminus\{i,j\}} p_{i\ell}P_\ell(H_i < \infty) \\ &= p_{ii} + p_{ij}f_{ji} + \sum_{\ell\in S\setminus\{i,j\}} p_{i\ell}f_{\ell i} \leqslant \sum_{\ell\neq j} p_{i\ell} < 1. \end{aligned}$$

□

Theorem 1.10. *Finite closed communicating classes are positive recurrent.*

Proof. Let $\mathcal{C}$ be the finite closed communicating class, then $\sum_{j\in\mathcal{C}} p_{ij}^{(n)} = 1$ for each $i \in \mathcal{C}$ and all $n \in \mathbb{N}_0$. If the class $\mathscr{C}$ was transient or null recurrent, then $\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = 0$ for all $j \in \mathcal{C}$. Hence, for transient or null recurrent and finite $\mathcal{C}$, we have

$$0 = \sum_{j\in\mathcal{C}} \lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)} = \lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathcal{C}} p_{ij}^{(k)} = 1.$$

This is a contradiction, hence $\mathcal{C}$ must be positive recurrent. □

Corollary 1.11. *An irreducible Markov chain on a finite state space is positive recurrent.*

For positive recurrence, we must focus only on Markov chain restricted to closed communicating classes. Hence, it suffices to study irreducible Markov chains.