Lecture-27: Poisson Process

1 Poisson and exponential random variables

A whole number valued random variable $N \in \mathbb{N}_0$ is called **Poisson** if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}.$$

It is easy to check that $\mathbb{E}N = \mathrm{Var}N = \lambda$. Furthermore, the moment generating function $M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t-1)}$ exists for all $t \in \mathbb{R}$.

1.1 Memoryless distribution

A random variable X with continuous support on $\mathbb{R}_+$, is called **memoryless** if

$$P\{X > s\} = P\{X > t+s | X > t\} \text{ for all } t, s \in \mathbb{R}_+.$$

Proposition 1.1. *The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.*

Proof. Let X be a random variable with a memoryless distribution function $F : \mathbb{R}_+ \to [0,1]$. It follows that $\bar{F}(t) \triangleq 1 - F(t)$ satisfies the semi-group property

$$\bar{F}(t+s) = \bar{F}(t)\bar{F}(s).$$

Since $\bar{F}(x) = P\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $\bar{F}(x) = e^{\theta x}$, for some $\theta < 0$ from Lemma A.1. $\square$

2 Simple point processes

A **simple point process** is a collection of distinct points $\Phi = \{S_n \in \mathbb{R}^d : n \in \mathbb{N}\}$, such that $|S_n| \to \infty$ as $n \to \infty$. Let $N(\emptyset) = 0$ and denote the number of points in a set $A \subseteq \mathbb{R}^d$ by $N(A) = \sum_{n\in\mathbb{N}} 1\{S_n \in A\}$. Then $(N(A) : A \in \mathcal{F})$ is called a **counting process** for the point process Φ. A counting process is **simple** if the underlying process is simple.

Point processes can model many interesting physical processes.

1. Aarrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.

2.1 Simple point processes in one-dimension

We can simplify this definition for $d = 1$. In $\mathbb{R}_+$, one can order the points $(S_n : n \in \mathbb{N})$ of the point process Φ. The number of points in the interval $(0,t]$ is $N((0,t]) = \sum_{n\in\mathbb{N}} 1\{S_n \in (0,t]\}$ as denoted by $N(t)$. For $s < t$, the number of points in interval $(s,t]$ is $N((s,t]) = N((0,t]) - N((0,s]) = N(t) - N(s)$.

A stochastic process $(N(t) : t \geqslant 0)$ is a **counting process** if

1. $N(0) = 0$, and
2. for each $\omega \in \Omega$, the map $t \mapsto N(t)$ is non-decreasing, integer valued, and right continuous.

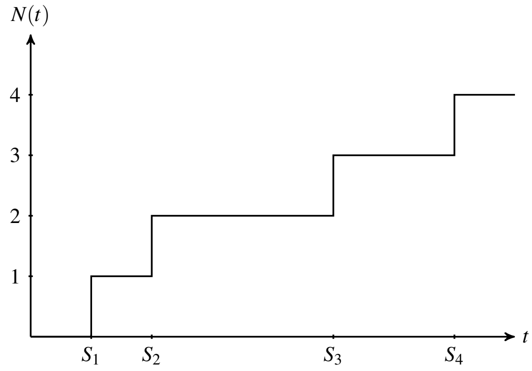

Figure 1: Sample path of a simple counting process.

Each discontinuity of the sample path of the counting process can be thought of as a jump of the process, as shown in Figure 1. A simple counting process has the unit jump size almost surely. General point processes in higher dimension don't have any inter-arrival time interpretation.

Lemma 2.1. *A counting process has finitely many jumps in a finite interval* $(0,t]$.

The points of discontinuity are also called the arrival instants of the point process $N(t)$. The n**th arrival instant** is a random variable denoted S_n, such that

$$S_0 = 0, \qquad\qquad S_n = \inf\{t \geq 0 : N(t) \geqslant n\},\ n \in \mathbb{N}.$$

The **inter arrival time** between $(n-1)$th and nth arrival is denoted by X_n and written as $X_n = S_n - S_{n-1}$. For a simple point process, we have

$$P\{X_n = 0\} = P\{X_n \leqslant 0\} = 0.$$

Lemma 2.2. *Simple counting process* $(N(t), t \geqslant 0)$ *and arrival process* $(S_n : n \in \mathbb{N})$ *are inverse processes, i.e.*

$$\{S_n \leqslant t\} = \{N(t) \geqslant n\}.$$

Proof. Let $\omega \in \{S_n \leqslant t\}$, then $N(S_n) = n$ by definition. Since N is a non-decreasing process, we have $N(t) \geq N(S_n) = n$. Conversely, let $\omega \in \{N(t) \geqslant n\}$, then it follows from definition that $S_n \leq t$. □

Corollary 2.3. *The following identity is true.*

$$\{S_n \leqslant t, S_{n+1} > t\} = \{N(t) = n\}.$$

Proof. It is easy to see that $\{S_{n+1} > t\} = \{S_{n+1} \leqslant t\}^c = \{N(t) \geqslant n+1\}^c = \{N(t) < n+1\}$. Hence,

$$\{N(t) = n\} = \{N(t) \geqslant n, N(t) < n+1\} = \{S_n \leqslant t, S_{n+1} > t\}.$$

□

Lemma 2.4. *Let $F_n(x)$ be the distribution function for S_n, then $P_n(t) \triangleq P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$.*

Proof. It suffices to observe that following is a union of disjoint events,

$$\{S_n \leqslant t\} = \{S_n \leqslant t, S_{n+1} > t\} \cup \{S_n \leqslant t, S_{n+1} \leqslant t\}.$$

□

3 Poisson process

A simple counting process $(N(t) : t \geqslant 0)$ is called a homogeneous **Poisson process** with a finite positive rate λ, if the inter-arrival times $(X_n : n \in \mathbb{N})$ are *iid* random variables with an exponential distribution of rate λ. That is, it has a distribution function $F : \mathbb{R}_+ \to [0,1]$, such that $F(x) = 1 - e^{-\lambda x}$ for all $x \in \mathbb{R}_+$.

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N(t) = n\}$ for $n \in \mathbb{N}_0$. We need the following lemma that enables us to do that.

Lemma 3.1. *For any finite time $t > 0$, a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have

$$\lim_{n\to\infty}\frac{S_n}{n}=E[X_1]=\frac{1}{\lambda}\quad\text{a.s.}$$

Fix $t>0$ and we define a sample space subset $M=\{\omega\in\Omega:N(\omega,t)=\infty\}$. For any $\omega\in M$, we have $S_n(\omega)\leqslant t$ for all $n\in\mathbb{N}$. This implies $\limsup_n\frac{S_n}{n}=0$ and $\omega\notin\{\lim_n\frac{S_n}{n}=\frac{1}{\lambda}\}$. Hence, the probability measure for set M is zero. □

3.1 Distribution functions

Lemma 3.2. *Moment generating function of arrival times S_n is*

$$M_{S_n}(t)=\mathbb{E}[e^{tS_n}]=\begin{cases}\frac{\lambda^n}{(\lambda-t)^n}, & t<\lambda\\ \infty, & t\geqslant\lambda.\end{cases}$$

Lemma 3.3. *Distribution function of S_n is given by $F_n(t)\triangleq P\{S_n\le t\}=1-e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}$.*

Theorem 3.4. *Density function of S_n is Gamma distributed with parameters n and λ. That is,*

$$f_n(s)=\frac{\lambda(\lambda s)^{n-1}}{(n-1)!}e^{-\lambda s}.$$

Theorem 3.5. *For each $t>0$, the distribution of Poisson process $N(t)$ with parameter λ is given by*

$$P\{N(t)=n)\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Further, $\mathbb{E}[N(t)]=\lambda t$, explaining the rate parameter λ for Poisson process.

Proof. Result follows from density of S_n and recognizing that $P_n(t)=F_n(t)-F_{n+1}(t)$. □

Corollary 3.6. *Distribution of arrival times S_n is*

$$F_n(t)=\sum_{j\geq n}P_j(t),\qquad\qquad \sum_{n\in\mathbb{N}}F_n(t)=\mathbb{E}N(t).$$

Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n\in\mathbb{N}}F_n(t)=\mathbb{E}\sum_{n\in\mathbb{N}}1\{N(t)\geqslant n\}=\sum_{n\in\mathbb{N}}P\{N(t)\geqslant n\}=\mathbb{E}N(t).$$

□

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E}N(t)=\lambda t$, which is linearly increasing in time.

A Functions with semigroup property

Lemma A.1. *A unique non-negative right continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying the semigroup property*

$$f(t+s)=f(t)f(s),\ \textit{for all}\ t,s\in\mathbb{R}$$

is $f(t)=e^{\theta t}$, where $\theta=\log f(1)$.

Proof. Clearly, we have $f(0)=f^2(0)$. Since f is non-negative, it means $f(0)=1$. By definition of θ and induction for $m,n\in\mathbb{Z}^+$, we see that

$$f(m)=f(1)^m=e^{\theta m},\qquad\qquad e^{\theta}=f(1)=f(1/n)^n.$$

Let $q\in\mathbb{Q}$, then it can be written as $m/n,n\neq 0$ for some $m,n\in\mathbb{Z}^+$. Hence, it is clear that for all $q\in\mathbb{Q}^+$, we have $f(q)=e^{\theta q}$. either unity or zero. Note, that f is a right continuous function and is non-negative. Now, we can show that f is exponential for any real positive t by taking a sequence of rational numbers $(q_n:n\in\mathbb{N})$ decreasing to t. From right continuity of f, we obtain

$$f(t)=\lim_{q_n\downarrow t}f(q_n)=\lim_{q_n\downarrow t}e^{\theta q_n}=e^{\theta t}.$$

□