Lecture-28: Characterizations of Poisson Process

1 Simple counting processes

1.1 independent increments

Proposition 1.1 (Markov property). *A simple counting process with independent increments property satisfies the following Markov property, for $0 < s < t$ and $n \in \mathbb{N}_0$,*

$$P(N(t) = n|\mathcal{F}_s) = P(N(t) = n|\sigma(N(s))).$$

Proof. Let $N(s) = m \leqslant n$ for some $m \in \mathbb{N}_0$, without any loss of generality. From the independence of the increments, we know that $N(t) - N(s)$ is independent of $\mathcal{F}_s$, and hence

$$P(N(t) = n|\mathcal{F}_s) = P(N(t) - N(s) = n - N(s)|\mathcal{F}_s) = P(N(t) - N(s) = n - N(s)|\sigma(N(s))) = P(N(t) = n|\sigma(N(s))).$$

□

From the definition of stopping times, for any stopping time τ of the counting process $(N(t) : t \geqslant 0)$, we have $\{\tau \leqslant t\} \in \mathcal{F}_t$. For counting processes $(N(t) : t \geqslant 0)$ with independent increments, we have $\{\tau \leqslant t\}$ independent of increments $(N(t+s) - N(t) : s \geqslant 0)$. One can check that the jump instants $(S_n : n \in \mathbb{N})$ are almost surely finite stopping times for $\lambda \in (0, \infty)$.

Theorem 1.2 (Strong Markov property). *Let τ be an almost surely finite stopping time of a simple counting process $(N(t) : t \geqslant 0)$ with independent increment property. Then, $(N(\tau + s) - N(\tau) : s \geqslant 0)$ is independent of the the stopping-time σ-algebra $\mathcal{F}_\tau$.*

1.2 stationary and independent increments

Lemma 1.3. *An arrival process $(S_n : n \in \mathbb{N}_0)$ has stationary and independent increments iff the sequence of inter-arrival times $(X_n : n \in \mathbb{N})$ are iid random variables.*

Proof. We first suppose that $(X_n : n \in \mathbb{N})$ is a sequence of *iid* random variables. Then $S_{n+m} - S_m$ has the same distribution as S_n and is independent of $(X_1, \dots, X_m)$. Conversely, we suppose that $(S_n : n \in \mathbb{N}_0)$ has stationary and independent increments. Then, $(X_n : n \in \mathbb{N})$ is a sequence of *iid* random variables by looking at $X_n = S_n - S_{n-1}$. □

Lemma 1.4. *If a simple counting process $(N(t), t \geqslant 0)$ has stationary and independent increments then the sequence of inter-arrival times $(X_n : n \in \mathbb{N})$ are iid random variables.*

Proof. To show that inter-arrival times are independent, it suffices to show that X_n is independent of S_{n-1}. First, we notice that from inverse relationship, we have

$$\{X_n > y\} = \{N(S_{n-1}) \leqslant N(S_{n-1} + y) < N(S_n) = N(S_{n-1}) + 1\} = \{N(S_{n-1} + y) - N(S_{n-1}) = 0\}.$$

From strong Markov property of a simple counting process with independent increments, we have $N(S_{n-1} + y) - N(S_n)$ independent of $N(S_{n-1})$, and hence

$$P\{S_{n-1} \leqslant x, X_n > y\} = P\{S_{n-1} \leqslant x, N(S_{n-1} + y) - N(S_{n-1}) = 0\} = P\{X_n > y\}F_{n-1}(x).$$

From stationarity of increments for the simple counting process $N(t)$, it follows that the distribution of $N(S_{n-1} + y) - N(S_{n-1})$ has same distribution as $N(y)$. Hence, we have each inter-arrival time is identically distributed,

$$P\{S_n - S_{n-1} > y|\mathcal{F}_{S_{n-1}}\} = P\{N(y) = 0\} = P\{X_1 > y\}.$$

□

Proposition 1.5. *Let $(N(t) : t \geqslant 0)$ be a simple counting process with stationary and independent increments, then $(N(t) : t \geqslant 0)$ is a homogeneous Poisson process.*

Proof. It suffices to show that X_1 is exponentially distributed. To this end, we show that the tail distribution $\bar{F}(t) \triangleq P\{X_1 > t\}$ of random variable X_1 is right continuous on $t \in \mathbb{R}_+$, and satisfies the semi-group property. Right continuity follows from right continuity of the counting process $N(t)$ and monotone convergence theorem. We observe that the following equality $\{X_1 > t\} = \{N(t) = 0\}$, and independent and stationary increment property of $N(t)$ to write for $t, s \in \mathbb{R}_+$

$$\bar{F}(t+s) = P\{N(t+s) = 0\} = P\{N(s) = 0\}P\{N(t+s) - N(s) = 0\} = \bar{F}(s)\bar{F}(t).$$

□

1.3 Age and excess time

At any time t, the instant of last and next arrivals for a simple point process are $S_{N(t)}$ and $S_{N(t)+1}$ respectively. For the associated simple counting process, the **age** is defined as the time since the last arrival, and the **excess** is defined as remaining time till next arrival. That is

$$A(t) = t - S_{N(t)} \qquad\qquad Y(t) = S_{N(t)+1} - t.$$

Lemma 1.6. *Age and residual processes for a Poisson process are independent and distributed identically to the inter-arrival times.*

Proof. We first find the distribution of age $A(s)$ and excess time $Y(s)$ individually. Using stationary increment property of the counting process $N(t)$, we can write

$$P\{A(s) > x\} = \sum_{n \in \mathbb{N}_0} P\{N(s) - N(s-x) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(x) = 0\}P\{N(s-x) = n\} = P_0(x),$$
$$P\{Y(s) > y\} = \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y) = 0\}P\{N(s) = n\} = P_0(y).$$

Since the counting process $N(t)$ has stationary and independent increments, we can write the joint probability as

$$\begin{aligned} P\{A(s) > x, Y(s) > y\} &= \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s-x) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y+x) = 0\}P\{N(s-x) = n\} \\ &= P\{N(y+x) = 0\} = P\{N(y+x) - N(y) = 0\}P\{N(y) = 0\} = P_0(x)P_0(y). \end{aligned}$$

Therefore, $Y(s)$ is independent of $A(s)$ and they both have the same exponential distribution as X_{n+1}. The memoryless property of exponential distribution is crucially used. □

2 Characterizations of Poisson process

It is clear that s partitions $X_{N(s)+1}$ in two parts such that $X_{N(s)+1} = A(s) + Y(s)$. It can be seen in Figure 1 for the case when $N(s) = n$.

Proposition 2.1. *A Poisson process $(N(t), t \geqslant 0)$ is simple counting process with stationary independent increments.*

Proof. Poisson process is a simple counting process by definition. To show that $N(t)$ has stationary and independent increments, it suffices to show that $N(t) - N(s)$ is independent of $N(s)$ and the distribution of increment $N(t) - N(s)$ is identical to that of $N(t-s)$. This follows from the fact that we can use induction to show stationary and independent increment property for for any finite disjoint time-intervals.

We can write the joint distribution of $N(t) - N(s)$ and $N(s)$ in terms of the following events involving inter-arrival times and excess times as

$$P\{N(t) - N(s) \geqslant m, N(s) = n\} = P\{Y(s) + S_{n+m} - S_{n+1} \leqslant t - s, S_n + A(s) = s\}.$$

Since the collection $(X_i : i \geqslant n+2) \cup \{Y(s)\}$ is independent of $(X_i : i \leqslant n) \cup A(s)$, we have $N(t) - N(s)$ independent of $N(s)$. We see that the increments are independent only if inter-arrival times are exponential. Further, since $Y(s)$ has same distribution as X_{n+1}, we get $N(t) - N(s)$ having same distribution as $N(t-s)$. □

Theorem 2.2 (Characterization 1). *The following are equivalent for a simple counting process $N = (N(t) : t \geqslant 0)$.*

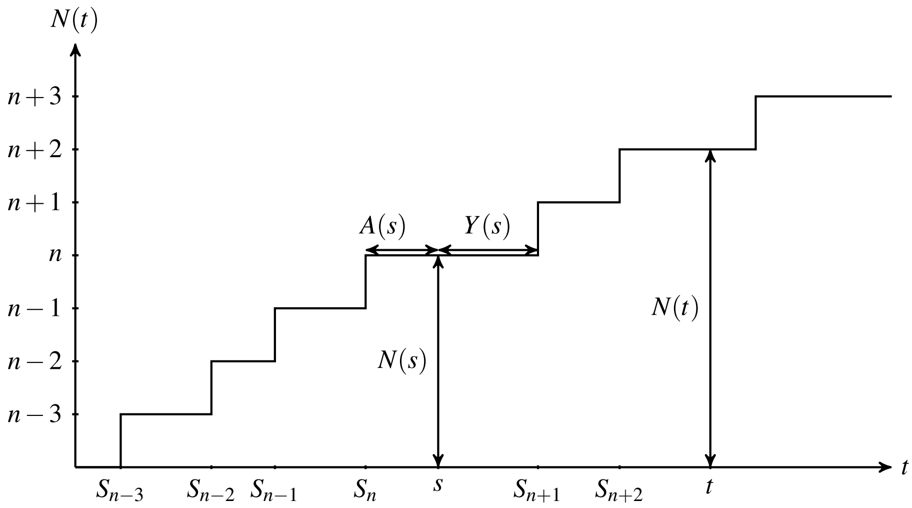

Figure 1: Stationary and independent increment property of Poisson process.

(a) Process N is Poisson with rate λ.

(b) Process N has independent increments, and the random variable $N(t)-N(s)$ is Poisson with mean $\lambda(t-s)$ for all $0 \leqslant s < t$.

(c) Process N has stationary and independent increments, and

$$\lim_{t \downarrow 0} \frac{P\{N(t)=1\}}{t} = \lambda, \qquad \lim_{t \downarrow 0} \frac{P\{N(t) \geq 2\}}{t} = 0.$$

Proof. We will show that $(a) \implies (b), (b) \implies (c)$, and $(c) \implies (a)$.

1. From Proposition 2.1, we have the first implication.
2. Stationarity is implied by the hypothesis in (b). Limits can be evaluated using the Poisson distribution.
3. It suffices to show that the rate of exponentially distributed first inter-arrival time X_1 is λ, which follows from the first two limits.

□

Theorem 2.3 (Characterization 2). *Let $\{I_i \subseteq \mathbb{R}_+ : i \in [k]\}$ be a finite collection of disjoint intervals. A stationary and independent increment simple counting process $(N(t) : t \geqslant 0)$ with $N(0)=0$ is Poisson process iff*

$$P \bigcap_{i=1}^{k} \{N(I_i) = n_i\} = \prod_{i=1}^{k} \frac{(\lambda |I_i|)^{n_i}}{n_i!} e^{-\lambda |I_i|}.$$

Proof. It is clear that Poisson process satisfies the above conditions. Further, since $P\{N(t)=0\} = e^{-\lambda t}$, it follows that the counting process with stationary and independent increment is Poisson with rate λ. □

Proposition 2.4. *Let $\{N(t), t \geqslant 0\}$ be a Poisson process with $\{I_i \subseteq \mathbb{R}_+ : i \in [n]\}$ a set of finite disjoint intervals with $I = \cup_{i \in [n]} I_i$, and $(k_i \in \mathbb{N}_0 : i \in [n])$ and $k = \sum_{i \in [n]} k_i$. Then, we have*

$$P(\cap_{i \in [n]} \{N(I_i) = k_i\} | \{N(I) = k\}) = k! \prod_{i \in [n]} \frac{1}{k_i!} \left(\frac{|I_i|}{|I|} \right)^{k_i}.$$

Proof. It follows from the stationary and independent increment property of Poisson processes that

$$P\{N(I_i) = k_i, i \in [n] | N(I) = k\} = \frac{P \bigcap_{i \in [n]} \{N(I_i) = k_i\}}{P\{N(I) = k\}} = \frac{\prod_{i \in [n]} P\{N(I_i) = k_i\}}{P\{N(I) = k\}}.$$

□