Lecture-29: Properties of Poisson Process

1 Conditional distribution of arrivals

Proposition 1.1. *For a Poisson process $(N(t) : t \geqslant 0)$, distribution of first arrival instant S_1 conditioned on the event $\{N(t) = 1\}$ is uniform between $(0,t]$.*

Proof. If $N(t) = 1$, then we know that conditional distribution of S_1 is supported on $(0,t]$. From independent increment property of the Poisson process $N(t)$, we have

$$P\{S_1 \leqslant s, N(t) = 1\} = P\{N(s) = 1\}P\{N(t) - N(s) = 0\}1_{\{s<t\}} + P\{N(t) = 1\}1_{\{s \geqslant t\}}.$$

The result follows from the stationarity of Poisson process, definition of conditional probability, and the Poisson distribution of $N(t)$. In particular, since $P\{N(t) = 1\} = e^{-t\lambda}\lambda t$ and $P\{N(t) = 0\} = e^{-t\lambda}$, we have

$$P(S_1 \leqslant s | N(t) = 1) = \frac{s}{t} 1_{\{s<t\}} + 1_{\{s \geqslant t\}}.$$

□

Proposition 1.2. *For a Poisson process $(N(t) : t \geqslant 0)$, joint distribution of arrival instant $\{S_1, \dots, S_n\}$ conditioned on $\{N(t) = n\}$ is identical to joint distribution of order statistics of n iid uniformly distributed random variables between $(0,t]$.*

Proof. Let $I_0 = \{0\}$ and $I_i \subset (0,t]$ be intervals such that $|I_i| = h_i$ and $\max I_{i-1} < \min I_i$ for each $i \in [n]$. Hence,

$$\bigcap_{i=1}^{n} \{S_i \in I_i\} \cap \{N(t) = n\} = \bigcap_{i=1}^{n} \{N(I_i) = 1\} \cap \{N((0,t] \setminus I) = 0\}.$$

The intervals I_i and $(0,t] \setminus I$ are disjoint. Hence from the independent and stationary increment property of the Poisson process $N(t)$, we get the probability of the above event as

$$P(\bigcap_{i=1}^{n} \{S_i \in I_i\} \cap \{N(t) = n\}) = \Big(\prod_{i=1}^{n} \lambda h_i e^{-\lambda h_i}\Big) e^{-\lambda(t - \sum_{i=1}^{n} h_i)} = \lambda^n e^{-\lambda t} \prod_{i=1}^{n} h_i.$$

Since $P\{N(t) = n\} = \exp(-\lambda t)\frac{(\lambda t)^n}{n!}$, it follows that $P\{S_1 \in I_1, \dots, S_n \in I_n | N(t) = n\} = n! \prod_{i=1}^{n} \frac{h_i}{t}$. Let $s_0 = 0 < s_1 < \dots < s_n \leqslant t$ and $h_i < s_i - s_{i-1}$ for each $i \in [n]$. Then $I_i = (s_i - h_i, s_i]$ are disjoint intervals of widths h_i, and we can find the joint density of $(S_1, \dots, S_n)$ conditioned on $\{N(t) = n\}$ as

$$f_{S_1,\dots,S_n | N(t)=n}(s_1, \dots, s_n) = \lim_{h_1,\dots,h_n \downarrow 0} \frac{1}{\prod_{i=1}^{n} h_i} n! \prod_{i=1}^{n} \frac{h_i}{t} = \frac{n!}{t^n}.$$

Let $U_1, \dots, U_n$ be *iid* uniform random variables in $[0,t]$. Then, the order statistics of $U_1 \dots, U_n$ has an identical joint distribution to n Poisson arrival instants conditioned on $\{N(t) = n\}$. □

2 Superposition and decomposition of Poisson processes

2.1 Merging

Let $(N_1(t) : t \geqslant 0)$ and $(N_2(t) : t \geqslant 0)$ be two independent Poisson processes. Then, the merged process of the two Poisson processes N_1, N_2 is denoted by N and point-wise defined as $N(t) = N_1(t) + N_2(t)$.

Theorem 2.1 (Superposition of independent processes). *A merged process of two Poisson processes with rates λ_1 and λ_2 is also Poisson with rate $\lambda = \lambda_1 + \lambda_2$.*

Proof. We show that the superposed process is a simple counting process with stationary and independent increments, and $P\{N(t)=0\}=e^{-\lambda t}$. Since $N_1(0)=0$ and $N_2(0)=0$, we have $N(0)=0$. Further, sum of two right-continuous, non-decreasing, integer-valued process remains right-continuous, non-decreasing, and integer-valued. Let S_k^i be the kth arrival instant of ith independent Poisson process. For simplicity, we show that $P\cup_{n,m\in\mathbb{N}}\{S_n^1=S_m^2\}=0$. Since this is a countable union of disjoint sets, it suffices to show that $P\{S_n^1=S_m^2\}=0$ for each $n,m\in\mathbb{N}$. However, that hold true since S_n^1,S_m^2 are independent continuous random variables.

For two disjoint intervals I_1,I_2 the number of arrivals for the superposed process are $N_1(I_1)+N_2(I_1)$ and $N_1(I_2)+N_2(I_2)$. Number of arrivals in disjoint intervals are independent, and hence $N_i(I_1)$ and $N_i(I_2)$ are independent for each $i\in\{1,2\}$. Further, the individual processes N_1,N_2 are independent and hence the increments are independent. To show the stationary increment property of the merged process, we take disjoint intervals $I_i\subset\mathbb{R}_+$ and $k_i\in\mathbb{N}_0$ for each $i\in[r]$. Then,

$$P\bigcap_{i=1}^{r}\{N(I_i)=k_i\}=P\bigcap_{i=1}^{r}\bigcup_{m_i+n_i=k_i}\{N_1(I_i)=m_i,N_2(I_i)=n_i\}=\prod_{i=1}^{r}\sum_{m_i+n_i=k_i}P\{N_1(I_i)=m_i\}P\{N_2(I_i)=n_i\}$$
$$=\prod_{i=1}^{r}\sum_{m_i+n_i=k_i}e^{-\lambda_1|I_i|}\frac{(\lambda_1|I_i|)^{m_i}}{m_i!}e^{-\lambda_2|I_i|}\frac{(\lambda_2|I_i|)^{n_i}}{n_i!}=\prod_{i=1}^{r}\frac{(\lambda|I_i|)^{k_i}}{k_i!}e^{-\lambda|I_i|}\sum_{m_i=0}^{k_i}\binom{k_i}{m_i}\left(\frac{\lambda_1}{\lambda}\right)^{m_i}\left(\frac{\lambda_2}{\lambda}\right)^{k_i-m_i}$$

Recognizing that the last summation is binomial expansion of $(\lambda_1+\lambda_2)^n/\lambda^n$, we get the stationarity. Further, taking disjoint intervals I_i such that $\cup_{i=1}^{r}I_i=(0,t]$, we get the Poisson distribution for the merged process. $\square$

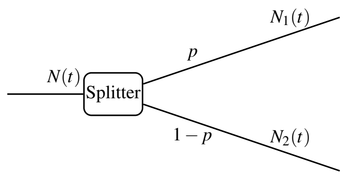
Remark 1. If the two processes are not independent, then the merged process is not necessarily Poisson.

2.2 Thinning

Consider a simple counting process $(N(t):t\geqslant 0)$ with rate λ and jump instants $(S_n:n\in\mathbb{N})$. Consider an independent Bernoulli process $(Z_n\in\{0,1\}:n\in\mathbb{N})$ independent of the simple counting process $N(t)$ and $\mathbb{E}Z_n=p$ for each $n\in\mathbb{N}$. We can split each incoming arrival at instant S_n to two streams 1 and 2, depending on whether $Z_n=1$ or 0 respectively. Correspondingly, we can define two split counting processes $(N_1(t):t\geqslant 0)$ and $(N_2(t):t\geqslant 0)$ such that

$$N_1(t)=\sum_{n\in\mathbb{N}}Z_n 1\{S_n\leqslant t\},\qquad\qquad N_2(t)=\sum_{n\in\mathbb{N}}\bar{Z}_n 1\{S_n\leqslant t\}.$$

It is easy to see that the split processes are also simple counting processes with $N_1(0)=N_2(0)=0$ and $N(t)=N_1(t)+N_2(t)$.



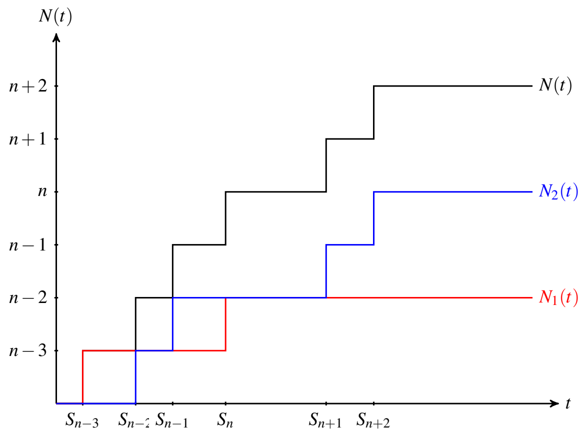


Figure 1: Splitting a Poisson process into two independent Poisson processes.

Theorem 2.2 (Independent splitting). *Let $(N(t), t \geqslant 0)$ be a Poisson process, and $(Z_n : n \in \mathbb{N})$ and iid Bernoulli sequence independent of the counting process. Then, two split counting processes $(N_1(t) : t \geqslant 0)$,and $(N_2(t) : t \geqslant 0)$ are mutually independent Poisson processes with rates $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1-p)$ respectively.*

Proof. Let $m \in \mathbb{N}$ and I_i be disjoint intervals and $k_i, \ell_i \in \mathbb{N}_0$ for each $i \in [m]$. For each interval I_i, we denote the arrival instants of Poisson process N falling in this interval by S_{ij}, and the corresponding Bernoulli random variables by Z_{ij}. Since I_i are disjoints, the collection of Bernoulli random variables in each interval are independent. Let $k_+\ell_i = m_i$, the joint finite dimensional distribution of two split processes are

$$P\bigcap_{i=1}^{m}\{N_1(I_i) = k_i, N_2(I_i) = \ell_i\} = P(\bigcap_{i=1}^{m}\{N(I_i) = n_i, \sum_{j=1}^{n_i} Z_{ij} = k_i\}) = \prod_{i=1}^{m} P\{N(I_i) = n_i, \sum_{j=1}^{k_i+\ell_i} Z_{ij} = k_i\}$$
$$= \prod_{i=1}^{m} e^{-\lambda|I_i|}\frac{(\lambda|I_i|)^{n_i}}{n_i!}\binom{n_i}{k_i}p^{k_i}(1-p)^{n_i-k_i} = \left(\prod_{i=1}^{m} e^{-\lambda_1|I_i|}\frac{(\lambda_1|I_i|)^{k_i}}{k_i!}\right)\left(\prod_{i=1}^{m} e^{-\lambda_2|I_i|}\frac{(\lambda_2|I_i|)^{\ell_i}}{\ell_i!}\right).$$

The result follows from second characterization of Poisson processes, and factorization of finite dimensional distributions of two split processes. □

A Order statistics

Let S_n be the symmetric group of all permutations on n elements. For any n length sequence $a \in \mathbb{R}^n$, the **order statistics** is a permutation $\sigma \in S_n$ such that

$$a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}.$$

For, $k \in [n]$, we call $a_{\sigma(k)}$ as the k**th order statistic** of the sequence a. In particular, first order statistic is the minimum, and the nth order statistic is the maximum of a n length sequence.

Lemma A.1. *Let $X = (X_1, X_2, \dots, X_n)$ be iid random variables with common density function f. Then, the joint density of order statistics of sequence X for a non-decreasing sequence $x \in \mathbb{R}^n$ is*

$$f_{X\circ\sigma}(x) = n!\prod_{i=1}^{n} f(x_i).$$

Proof. Let $x \in \mathbb{R}^n$ be a non-decreasing sequence. Since X is an *iid* sequence, we have $f_X(x) = \prod_{i=1}^{n} f(x_i)$. Further, for any permutation $\gamma \in S_n$, we have $f_X(x) = f_X(x \circ \gamma)$. The result follows since $\{X \circ \sigma = x\} = \cup_{\gamma \in S_n}\{X = x \circ \gamma\}$ and $|S_n| = n!$. □

Lemma A.2. *Let $X = (X_1, X_2, \dots, X_n)$ be iid random variables with common distribution function F. Then, the distribution function of kth order statistic of sequence X for $x \in \mathbb{R}$ is*

$$F_{X_{\sigma(k)}}(x) = \binom{n}{k} F(x)^k \bar{F}(x)^{n-k}.$$

Proof. For any $x \in R$, we can write the event

$$\{X_{\sigma(k)} \leqslant x\} = \bigcup_{S \subset [n]: |S| = k} \left\{\max_{i \in S} X_i \leqslant x, \min_{i \notin S} X_i \geqslant x\right\}$$

From *iid* nature of sequence X, it follows that each of the event inside the union has equal probability equal to

$$P\left\{\max_{i \in S} X_i \leqslant x, \min_{i \notin S} X_i \geqslant x\right\} = F(x)^k \bar{F}(x)^{n-k}.$$

The result follows from the fact that the number of events inside the union is $|\{S \subset [n] : |S| = k\}| = \binom{n}{k}$. □