Lecture-30: Compound and Non-Stationary Poisson Processes

1 Compound Poisson process

A **compound Poisson process** is a real-valued right-continuous process $(Z_t : t \geq 0)$ with the following properties.

i_ finite jumps: for all $\omega \in \Omega$, sampled path $t \mapsto Z_t(\omega)$ has finitely many jumps in finite intervals,

ii_ independent increments: for all $t,s \geq 0; Z_{t+s} - Z_t$ is independent of past $(Z_u : u \leq t)$,

iii_ stationary increments: for all $t,s \geq 0$, distribution of $Z_{t+s} - Z_t$ depends only on s and not on t.

For each $\omega \in \Omega$ and $n \in \mathbb{N}$, we can define time and size of nth jump

$$S_0(\omega) = 0, \qquad S_n(\omega) = \inf\{t > S_{n-1} : Z_t(\omega) \neq Z_{S_{n-1}}(\omega)\}$$
$$X_0(\omega) = 0, \qquad X_n(\omega) = Z_{S_n}(\omega) - Z_{S_{n-1}}(\omega).$$

Let $(N_t : t \geqslant 0)$ be the simple counting process associated with the number of jumps in $(0,t]$. Then, S_n and X_n are the respectively the arrival instant and the size of the nth jump, and we can write $Z_t = \sum_{i=1}^{N_t} X_i$. Let $\mathcal{F}_s = \sigma(Z_u : u \in (0,s])$, and $\mathcal{F}_\bullet = (\mathcal{F}_s : s \geqslant 0)$ be the natural filtration associated with the process Z. Clearly, jump times $(S_n : n \in \mathbb{N})$ are stopping times with respect to filtration $\mathcal{F}_\bullet$.

Proposition 1.1. *A stochastic process $(Z_t, t \geqslant 0)$ is a compound Poisson process iff its jump times form a Poisson process and the jump sizes form an iid random sequence independent of the jump times.*

Proof. It is clear that the simple counting process N_t can be completely determined by $(Z_u : u \leqslant t)$, i.e. $N_t \in \mathcal{F}_t$. Since $Z_{t+s} - Z_t = \sum_{i=N_t}^{N_{t+s}} X_i$, and the compound Poisson processes have independent increments, it follows that the increment $(N_{t+s} - N_t : s \geqslant 0)$ and $(X_{N(t)+j} : j \in \mathbb{N})$ are independent of the past $\mathcal{F}_t$. Let's assume that step sizes are positive, then $S_n = \inf\{t > S_{n-1} : Z_t > Z_{S_{n-1}}\}$ and $\{N_{t+s} - N_t = 0\} = \{Z_{t+s} - Z_t \leqslant 0\}$. From the stationarity of the increments it follows that the probability $P\{N_{t+s} - N_t = 0\}$ is independent of t, and hence $(N_t : t \geqslant 0)$ is a Poisson process, and that stopping time in the process $(S_n : n \in \mathbb{N})$ is almost surely finite.

The compound Poisson process has the Markov property from stationary and independent increment property. Further, since each sample path $t \mapsto Z_t$ is right continuous, the process satisfies the strong Markov property at each almost sure stopping time. In particular, X_n is independent of the past $(Z_u : u \leqslant S_{n-1})$ and identically distributed to X_1 for each $n \in \mathbb{N}$. It follows that the jump sizes $X_1, X_2, \dots$ are *iid* random variables, independent of arrival instants $S_1, S_2, \dots$. Similar arguments can be used to show for negative jump sizes. For real jump sizes, we can form two independent Poisson processes with negative and positive jumps, and the superposition of these two processes is Poisson.

Conversely, let $Z_t = \sum_{i=1}^{N_t} X_i$ where N_t is a Poisson process independent of the random *iid* sequence $X_1, X_2, \dots$. Since N_t is finite for any finite t, it follows that the compound Poisson process Z has finitely many jumps in finite intervals. For any finite $n \in \mathbb{N}$ and finite intervals I_i for $i \in [n]$, we can write $Z(I_i) = \sum_{k=1}^{N(I_i)} X_{ik}$, where X_{ik} denotes the kth jump size in the interval I_i. Since the independent sequence $(N(I_i) : i \in [n])$ and $(X_n : n \in \mathbb{N})$ are also mutually independent, it follows that $Z(I_i)$ are independent. Further, the stationarity of the increments of the compound process is inferred from the distribution of $Z(I_i)$, which is

$$P\{Z(I_i) \leqslant x\} = \sum_{m \in \mathbb{N}_0} P\{Z(I_i) \leqslant x, N(I_i) = m\} = \sum_{m \in \mathbb{N}_0} P\{\sum_{k=1}^{m} X_{ik} \leqslant x\} P\{N(I_i) = m\}$$

□

Example 1.2. Examples of compound Poisson processes.

- Arrival of customers in a store is a Poison process N_t. Each customer i spends an *iid* amount X_i independent of the arrival process.

$$Y_0 = 0, \qquad\qquad Y_n = \sum_{i=1}^{n} X_i, i \in [n].$$

Now define $Z_t = Y_{N_t}$ as the amount spent by the customers arriving in time t. Then $\{Z_t, t \geq 0\}$ is a compound Poisson Process.

- Let the time between successive failures of a machine be independent and exponentially distributed. The cost of repair is *iid* random at each failure. Then the total cost of repair in a certain time t is a compound Poisson Process.

2 Non-stationary Poisson process

From the characterization of Poisson process just stated, we can generalize to non-homogeneous Poisson Process. In this case, the rate of Poisson Process λ is time varying. A simple counting process $(N(t) : t \geqslant 0)$ is said to be possibly **non-stationary Poisson process** if $N(0) = 0$ and it has independent increments. That is,

1. for each $\omega \in \Omega$, each sample path $t \mapsto N_t(\omega)$ is right continuous, non-decreasing, integer valued, $N(0) = 0$, and has jumps of unit size only,
2. for any $t, s \geqslant 0$, the random variable $N_{t+s} - N_t$ is independent of the past $(N_u : u \leqslant t)$.

Let $m(t) = \mathbb{E}N_t$ for all $t \geqslant 0$. From non-decreasing property of counting processes, it follows that the mean is also non-decreasing in time t. From right continuity of counting process and the monotone convergence theorem, it follows that mean function is also right continuous. The **time inverse** of mean is defined as

$$\tau(t) = \inf\{s > 0 : m(s) > t\},\ t \geqslant 0.$$

It follows that the following events are identical, $\{s > \tau(t)\} = \{m(s) > t\}$. Therefore, if m is a continuous function, we would have $m(\tau(t)) = t$. Since inverse of a non-decreasing function is also non-decreasing, we conclude that $\tau(t)$ is non-decreasing function of time t. We can also see that by taking $t_1 \leqslant t_2$. Then for all $s > \tau(t_2)$ we have $m(s) > t_2 \geqslant t_1$. In particular, we have $m(s) > \tau(t_1)$ and hence $\tau(t_2) \geqslant \tau(t_1)$. From the definition of $\tau(t)$, it follows that $\{s > \tau(m(t))\} = \{m(s) > m(t)\}$. Hence,

$$\tau(m(t)) = \sup\{s > 0 : m(s) \leqslant m(t)\}.$$

If m is differentiable for all $t > 0$, then we can define $\lambda(t) = m'(t)$ for all $s > 0$. In higher dimensions, we have $m(A) = \mathbb{E}N(A)$ for some Borel measurable set $A \subseteq \mathbb{R}^d$. If m is differentiable for all $x \in \mathbb{R}^d$, then for $A = \prod_{i=1}^{d}(x_i, x_i + h_i]$ we can write the intensity $\Lambda(x) = \lim_{h_1,\dots,h_d \to 0} \frac{m(A)}{h_1 \dots h_d}$. In other words, we have $m(A) = \mathbb{E}N(A) = \int_{x \in A} \Lambda(x)dx$.

Theorem 2.1. *Let N_t be a non-stationary Poisson process, such that $m(t) = \mathbb{E}N_t$ is continuous. Then, the process $M : \Omega \times \mathbb{R}_+ \to \mathbb{N}_0$ is a stationary Poisson process with unit rate, defined point-wise as*

$$M_t(\omega) \triangleq N_{\tau(t)}(\omega),\ \ t \geqslant 0, \omega \in \Omega.$$

Proof. Since τ is non-decreasing and continuous and N is a simple counting process, it follows that M is right continuous, non-decreasing, integer valued, and simple. Fix $t > s \geq 0$ and let $s' \triangleq \tau(s)$ and $t' \triangleq \tau(t)$. Then, by definition of M_t, t', s' and independent increment property of non-stationary Poisson process N, we have $M_t - M_s = N_{t'} - N_{s'}$ is independent of the past $(N_u : u \leqslant s') = (M_u : u \leqslant s)$. That is, the process M has independent increments. For stationarity, we see that mean rate of increment is unity and stationary,

$$\mathbb{E}[M_t - M_s | M_u; u \leqslant s] = \mathbb{E}[N_{t'} - N_{s'} | N_u; u \leqslant s'] = m(t') - m(s') = m(\tau(t)) - m(\tau(s)) = t - s.$$

□

Corollary 2.2. *Let $m(t)$ be a continuous non-decreasing function. Then, $S_1, S_2, \dots$ are the arrival instants in a non-stationary Poisson process N_t with mean function $m(t) = \mathbb{E}N_t$ iff $m(S_1), m(S_2), \dots$ are the arrivals instants of a stationary Poisson process of unit rate.*

Proof. We can write the nth arrival instant S'_n of unit-rate stationary Poisson process M_t, in terms of the nth arrival instant S_n of non-stationary Poisson process N_t as

$$S'_n = \inf\{t > 0 : \tau(t) \geqslant S_n\} = \sup\{t > 0 : m(S_n) \geqslant t\} = m(S_n).$$

$\square$

This corollary implies that $S_n \in (s,t]$ if and only if $m(S_n) \in (m(s), m(t)]$. Therefore, thr number of arrivals in $(s,t]$ equals the number of arrivals for unit-rate stationary Poisson process in $(m(s), m(t)]$. Hence, we conclude that for $m(s,t] = m(t) - m(s)$

$$P\{N_t - N_s = k\} = e^{-m(s,t]} \frac{m(s,t]^k}{k!}, \ k \in \mathbb{N}_0.$$

We will see that the inter-arrival times for the non-stationary Poisson process N_t, defined as

$$T_0 = 0, \qquad\qquad T_n = S_n - S_{n-1}, \ n \in \mathbb{N},$$

are not independent anymore.

Proposition 2.3. *For a non-stationary Poisson process with continuous mean function $m(t)$, we have*

$$P\{T_{n+1} > t | S_1, S_2, \dots, S_n\} = \exp\left(-m(S_n + t) + m(S_n)\right).$$

Proof. We define events $A = \{m(S_{n+1}) > m(S_n + t)\}$ and $B = \{m(S_{n+1}) \geq m(S_n + t)\}$. Then, we have $A \subseteq \{T_{n+1} > t\} \subseteq B$. Hence, we can write

$$P\{A | S_1, S_2, \dots, S_n\} \leq P\{T_{n+1} > t | S_1, S_2, \dots, S_n\}.$$

The arrival instants $S_1, \dots, S_n$ determine $m(S_1), \dots, m(S_n), m(S_n + t)$. Further, since $m(S_{n+1}) - m(S_n)$ is the inter-arrival time of the stationary Poisson process $M(t)$, it is independent of $S_1, S_2, \dots, S_n$

$$P\{A | S_1, \dots, S_n\} = P\{m(S_{n+1}) - m(S_n) > m(S_n + t) - m(S_n) | S_1, \dots, S_n\} = \exp\left(-m(S_n + t) + m(S_n)\right).$$

Result follows from the continuity of the exponential distribution. $\square$