Lecture-31: Poisson Point Processes

1 Poisson Point Processes

A simple point process $\Phi = \{S_i \in \mathbb{R}^d : i \in \mathbb{N}\}$ is a random countable collection of distinct points in the d-dimensional Euclidean space $\mathbb{R}^d$. Let $\mathcal{F}$ be the collection of Borel measurable sets in $\mathbb{R}^d$. Then, the associated counting process $(N(A) : A \in \mathcal{F})$ is defined as $N(A) = \sum_{i \in \mathbb{N}} 1_{\{S_i \in A\}}$. For a sample realization of a simple point process Φ, we observe that $dN(x) = 0$ for all $x \notin \Phi$. Hence, for any function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\int_{x \in A} f(x) dN(x) = \sum_{i \in \mathbb{N}} f(S_i) 1_{\{S_i \in A\}}.$$

The finite dimensional distribution of N is joint distribution of $(N(A_1), \dots, N(A_k))$ for some finite $k \in \mathbb{N}$ and bounded sets $A_1, \dots, A_k \in \mathcal{F}$. A counting process N has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{F}$,

$$P \bigcap_{i=1}^{k} \{N(A_i) = n_i\} = \prod_{i=1}^{n} P\{N(A_i) = n_i\}.$$

A **Poisson point process** of intensity measure Λ is defined in terms of the finite-dimensional distribution of the associated counting process, as

$$P \bigcap_{i=1}^{k} \{N(A_i) = n_i\} = \prod_{i=1}^{k} \left(e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^n_i}{n_i!} \right),$$

for some finite $k \in \mathbb{N}$, mutually disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{F}$ and $\Lambda(A) \triangleq \int_{x \in A} d\Lambda(x)$.

Theorem 1.1. *Following are equivalent for a simple counting process N.*

i_ Process N is Poisson with intensity measure Λ.

ii_ Process N has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$.

iii_ The intensity measure Λ is bounded for bounded $A \in \mathcal{F}$, and $N(A)$ is a Poisson with parameter $\Lambda(A)$.

Proposition 1.2. *For a finite $k \in \mathbb{N}$, disjoint bounded sets $A_1, \dots, A_k \in \mathcal{F}$, numbers $n_1, \dots, n_k \in \mathbb{N}_0$, let $A = \cup_{i=1}^k A_i$ and $n = \sum_{i=1}^k n_i$. Then for a Poisson point process, we have*

$$P(N(A_1) = n_1, \dots, N(A_k) = n_k | N(A) = n) = \frac{n!}{n_1! \dots n_k!} \prod_{i=1}^{n} \left(\frac{\Lambda(A_i)}{\Lambda(A)} \right)^{n_i}.$$

Remark 1. Above proposition implies that n points in a bounded set A are iid distributed in this set with density function $\frac{\Lambda'(x)}{\Lambda(A)}$ for $x \in A$.

2 Laplace functional

The **Laplace functional** $\mathcal{L}$ of a point process Φ and associated counting process N is defined for all non-negative function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathcal{L}_\Phi(f) = \mathbb{E} \exp\left(- \int_{\mathbb{R}^d} f(x) dN(x) \right).$$

For simple function $f(x) = \sum_{i=1}^k t_i 1\{x \in A_i\}$, we can write the Laplace functional

$$\mathcal{L}_\Phi(f) = \mathbb{E} \exp(-\sum_i t_i N(A_i)),$$

as a function of the vector $(t_1, t_2, \dots, t_k)$, a joint Laplace transform of the random vector $(N(A_1), \dots, N(A_k))$. This way, one can compute all finite dimensional distribution of the counting process N.

Proposition 2.1. *The Laplace functional of the Poisson process with intensity measure* Λ *is*

$$\mathcal{L}_\Phi(f) = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-f(x)})\Lambda(dx)\right).$$

Proof. For a bounded Borel measurable set $A \subseteq \mathbb{R}^d$, consider $g(x) = f(x)1\{x \in A\}$. Then,

$$\mathcal{L}_\Phi(g) = \mathbb{E}\exp(-\int_{\mathbb{R}^d} g(x)dN(x)) = \mathbb{E}\exp(-\int_A f(x)dN(x)).$$

Clearly $dN(x) = \delta_x 1\{x \in \Phi\}$ and hence we can write $\mathcal{L}_\Phi(g) = \mathbb{E}\exp\left(-\sum_{S_i\in\Phi\cap A} f(S_i)\right)$. We know that the probability of $N(A) = |\Phi(A)| = n$ points in set A is given by

$$P\{N(A) = n\} = e^{-\Lambda(A)}\frac{\Lambda(A)^n}{n!}.$$

Given there are n points in set A, the density of n point locations are independent and given by

$$f_{S_1,\dots,S_n}(x_1,\dots,x_n) = \prod_{i=1}^n \frac{\Lambda(dx_i)1\{x_i \in A\}}{\Lambda(A)}.$$

Hence, we can write the Laplace functional as

$$\mathcal{L}_\Phi(g) = e^{-\Lambda(A)}\sum_{n\in\mathbb{N}_0}\frac{\Lambda(A)^n}{n!}\prod_{i=1}^n\int_A e^{-f(x_i)}\frac{\Lambda(dx_i)}{\Lambda(A)} = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-g(x)})\Lambda(dx)\right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. □

2.1 Superposition of point processes

Theorem 2.2. *The superposition of independent Poisson point processes with intensities* Λ_k *is a Poisson point process with intensity measure* $\sum_k \Lambda_k$ *if and only if the latter is a locally finite measure.*

2.2 Thinning of point processes

Consider a probability **retention function** $p : \mathbb{R}^d \to [0,1]$ and a point process Φ. The **thinning** of point process $\Phi = \{S_n \in \mathbb{R}^d : n \in \mathbb{N}\}$ with the retention function p is a point process such that

$$\Phi^p = \{S_n \in \Phi : Y(S_n) = 1\},$$

where $Y(S_n)$ is an independent indicator stochastic process at each point S_n and $\mathbb{E}Y(S_n) = p(S_n)$.

Theorem 2.3. *The thinning of the Poisson point process of intensity measure* Λ *with the retention probability function* p *yields a Poisson point process of intensity measure* $p\Lambda$ *with*

$$(p\Lambda)(A) = \int_A p(x)\Lambda(dx)$$

for all bounded Borel measurable $A \subseteq \mathbb{R}^d$.

Proof. Let $A \subseteq \mathbb{R}^d$ be a bounded Boreal measurable set, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function. Let N^p be the associated counting process to the thinned point process Φ^p. Hence, for any set $A \in \mathcal{F}$, we have $N^p(A) = \sum_{i\in\Phi(A)} Y(S_i)$. Consider the Laplace functional of the thinned point process Φ^p for a non-negative function $g(x) = f(x)1\{x \in A\}$

$$\mathcal{L}_{\Phi^p}(g) = \mathbb{E}\mathbb{E}[\exp\left(-\int_A f(x)dN^p(x)\right)\Big|\Phi(A)] = \sum_{n\in\mathbb{N}_0} P\{N(A) = n\}\prod_{i=1}^n \mathbb{E}[\exp\left(-f(S_iY(S_i))\right)\Big|S_i \in \Phi(A)].$$

The first equality follows from the definition of Laplace functional and taking nested expectations. Second equality follows from the fact that the distribution of all points of a Poisson point process are iid. Since Y is a Bernoulli process independent of the underlying process Φ with $\mathbb{E}[Y(S_i)] = p(S_i)$, we get

$$\mathbb{E}[e^{-f(S_iY(S_i))}\Big|S_i \in \Phi(A)] = \mathbb{E}[e^{-f(S_i)}p(S_i) + (1-p(S_i))\Big|S_i \in \Phi(A)].$$

From the distribution $\frac{\Lambda'(x)}{\Lambda(A)}$ for $x \in \Phi(A)$ for the Poisson point process Φ, we get

$$\mathcal{L}_{\Phi^p}(g) = e^{-\Lambda(A)}\sum_{n\in\mathbb{N}_0}\frac{1}{n!}\left(\int_A(p(x)e^{-f(x)} + (1-p(x))\Lambda(dx)\right)^n = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-g(x)})p(x)\Lambda(dx)\right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. □