Lecture 02: Bernoulli Processes

1 Construction of Probability Space

Consider an experiment, where an infinite sequence of trials is conducted. Each trial has two possible outcomes, success or failure, denoted by S and F respectively. Any outcome of the experiment is an infinite sequence of successes and failures, e.g.

$$\omega = (S, F, F, S, F, S, \dots).$$

The collection of all possible outcomes of this experiment will be our sample space $\Omega = \{S, F\}^{\mathbb{N}}$. The ith projection of an outcome sequence $\omega \in \Omega$ is denoted by $\omega_i \in \{S, F\}$. We consider a σ-algebra $\mathcal{F}$ on this space generated by all finite subsets of the sample space Ω. That is,

$$\mathcal{F} = \sigma(\{\omega \in \Omega : \omega_i \in \{S, F\}, \forall i \in I \subset \mathbb{N} \text{ for finite } I\}).$$

We further assume that each trial is independent and identically distributed, with common distribution of a single trial

$$P\{\omega_i = S\} = p, \qquad P\{\omega_i = F\} = q \triangleq 1 - p.$$

This assumption completely characterizes the probability measure over all elements of the σ-algebra $\mathcal{F}$. For $a \in \mathcal{F}$ and the number of successes $n = |\{i \in I : a_i = S\}|$ in I,

$$P(a) = \prod_{i \in I} \mathbb{E}1\{\omega_i = a_i\} = \prod_{i \in I : \omega_i = S} \mathbb{E}1\{\omega_i = S\} \prod_{i \in I : \omega_i = F} \mathbb{E}1\{\omega_i = F\} = p^n q^{|I| - n}.$$

Hence, we have completely characterized the probability space $(\Omega, \mathcal{F}, P)$. Further, we define a discrete random process $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ such that

$$X_n(\omega) = 1\{\omega_n = S\}.$$

Since, each trial of the experiment is iid, so is each X_n.

2 Bernoulli Processes

For a probability space $(\Omega, \mathcal{F}, P)$, a discrete process $X = \{X_n(\omega) : n \in \mathbb{N}\}$ taking value in $\{0, 1\}^{\mathbb{N}}$ is a **Bernoulli Process** with success probability $p = \mathbb{E}X_n$ if $\{X_n : n \in \mathbb{N}\}$ are iid with common distribution $P\{X_n = 1\} = p$ and $P\{X_n = 0\} = q$.

i. For products manufactured in an assembly line, X_n indicates the event of nth product being defective.

ii. At a fork on the road, X_n indicates the event of nth vehicle electing to go left on the fork.

For $n = |\{i \in S : 0 \leq x_i < 1\}|$, the finite dimensional distribution of $X(\omega)$ is given by

$$F_S(x) = \prod_{i \in S} P\{X_i \leq x_i\} = q^n.$$

The mean, correlation, and covariance functions are given by

$$m_X = \mathbb{E}X_n = p, \qquad R_X = \mathbb{E}X_n X_m = p^2, \qquad C_X = \mathbb{E}(X_n - p)(X_m - p) = 0.$$

3 Number of Successes

For the above experiment, let N_n denote the number of successes in first n trials. Then, we have

$$N_n(\omega) = \sum_{i=1}^{n} 1\{\omega_i = S\} = \sum_{i=1}^{n} X_i(\omega).$$

The discrete process $\{N_n(\omega) : n \in \mathbb{N}\}$ is a stochastic process that takes discrete values in $\mathbb{N}_0$. In particular, $N_n \in \{0, \ldots, n\}$, i.e. the set of all outcomes is index dependent.

i_ For products manufactured in an assembly line, N_n indicates the number of defective products in the first n manufactured.

ii_ At a fork on the road, N_n indicates the number of vehicles that turned left for first n vehicles that arrived at the fork.

We can characterize the moments of this stochastic process

$$m_N(n) = \mathbb{E}X_n = np, \qquad \operatorname{Var} N_n = \sum_{i=1}^{n} \operatorname{Var} X_i = npq.$$

Clearly, this process is not stationary since the first moment is index dependent. In the next lemma, we try to characterize the distribution of random variable N_n.

Lemma 3.1. *We can write the following recursion for $P_n(k) \triangleq P\{N_n(\omega) = k\}$, for all $n, k \in \mathbb{N}$*

$$P_{n+1}(k) = pP_n(k-1) + qP_n(k).$$

Proof. We can write using the disjoint union of probability events,

$$P\{N_{n+1} = k\} = P\{N_{n+1} = k, N_n = k\} + P\{N_{n+1} = k, N_n = k-1\}.$$

Since $\{N_{n+1} = k, N_n = j\} = \{X_{n+1} = k - j, N_n = j\}$, from independence of X_ns we have

$$P\{N_{n+1} = k\} = P\{X_{n+1} = 0\}P\{N_n = k\} + P\{X_{n+1} = 1\}P\{N_n = k-1\}.$$

Result follows from definition of $P_n(k)$. □

Theorem 3.2. *The distribution of number of successes N_n in first n trials of a Bernoulli process is given by a Binomial (n, p) distribution*

$$P_n(k) = \binom{n}{k} p^k q^{(n-k)}.$$

Proof. It follows from induction. □

Corollary 3.3. *The stochastic process* $\{N_n : n \in \mathbb{N}\}$ *has stationary and independent increments.*

Proof. We can look at one increment

$$N_{m+n} - N_m = \sum_{i=1}^{n} X_{i+m}.$$

This increment is a function of $(X_{m+1}, \dots, X_{m+n})$ and hence independent of $(X_1, \dots, X_m)$. The random variable N_m depends solely on $(X_1, \dots, X_m)$ and hence the independence follows. Stationarity follows from the fact that X_is are iid and $N_{m+n} - N_m$ is sum of n iid Bernoulli random variables, and hence has a Binomial (n, p) distribution identical to that of N_n. □

4 Times of Successes

For the above experiment, let T_k denote the trial number corresponding to kth success. Clearly, $T_k \geq k$. We can define it inductively as

$$T_1 = \inf\{n \in \mathbb{N} : X_n(\omega) = 1\}, \qquad T_{k+1}(\omega) = \inf\{n > T_k : X_n(\omega) = 1\}.$$

For example, if $X = (0, 1, 0, 1, 1, \dots)$, then $T_1 = 2, T_2 = 4, T_3 = 5$ and so on. The discrete process $\{T_k(\omega) : k \in \mathbb{N}\}$ is a stochastic process that takes discrete values in $\{k, k+1, \dots\}$.

i_ For products manufactured in an assembly line, T_k indicates the number of products inspected for kth defective product to be detected.

ii_ At a fork on the road, T_k indicates the number of vehicles that have arrived at fork for kth left turning vehicle.

We first observe the following inverse relationship between number of nth successful trial T_n, and number of successes in n trials.

Lemma 4.1. *The following relationships hold between time of success and number of success*

$$\{T_k \leq n\} = \{N_n \geq k\}, \qquad \{T_k = n\} = \{N_{n-1} = k-1, X_n = 1\}.$$

Proof. To see the first equality, we observe that $\{T_k \leq n\}$ is the set of outcomes, where $X_{T_1} = X_{T_2} = \cdots = X_{T_k} = 1$, and $\sum_{i=1}^{T_k} X_i = k$. Hence, we can write the number of successes in first n trials as

$$N_n = \sum_{i=1}^{n} X_i = \sum_{i > T_k} X_i + \sum_{i=1}^{T_k} X_i \geq k.$$

Conversely, we notice that we can re-write the number of trials for ith success as

$$T_i = \inf\{m \in \mathbb{N} : N_m = i\}.$$

Since N_n is non-decreasing in n, it follows that for the set of outcome such that $\{N_n \geq k\}$, there exists $m \leq n$ such that $T_k = m \leq n$. For the second inequality, we observe that

$$\{T_k = n\} = \{T_k \leq n\} \cap \{T_k \geq n-1\}^c = \{N_n \geq k\} \cap \{N_{n-1} \geq k\}^c = \{N_{n-1} = k-1, N_n = k\}.$$

□

We can write the marginal distribution of process $\{T_k : k \in \mathbb{N}\}$ in terms of the marginal of the process $\{N_n : n \in \mathbb{N}\}$ as

$$P\{T_k \leq n\} = \sum_{j \geq k} P_n(j) = \sum_{j=k}^{\infty} \binom{n}{j} p^j q^{(n-j)}, \quad P\{T_k = n\} = p\dot{P}_{n-1}(k-1) = \binom{n-1}{k-1} p^k q^{n-k}.$$

Clearly, this process is not stationary since the first moment is index dependent. It is not straightforward to characterize moments of T_k from its marginal distribution.

Lemma 4.2. *The time of success is a Markov chain.*

Proof. By definition, T_k depends only on T_{k-1} and X_is for $i > T_k$. From independence of X_is, it follows that

$$P\{T_k = T_{k-1} + m | T_{k-1}, T_{k-2}, \dots, T_1\} = P\{T_k - T_{k-1} = m | T_{k-1}\} = q^{m-1} p 1\{m > 0\}.$$

□

Corollary 4.3. *The time of success process $\{T_k : k \in \mathbb{N}\}$ has stationary and independent increments.*

Proof. From the previous lemma and the law of total probability, we see that

$$\sum_{n \in \mathbb{N}_0} P\{T_k - T_{k-1} = m, T_k = n\} = pq^{m-1} 1\{m > 0\}.$$

□

Since, the increment in time of success follows a geometric distribution with success probability p, we have mean of the increment $\mathbb{E}(T_{k+1} - T_k) = 1/p$, and the variance $\operatorname{Var} T_{k+1} - T_k = q/p^2$. This implies that we can write the moments of T_k as

$$\mathbb{E}T_k = \mathbb{E}\sum_{i=1}^{k}(T_i - T_{i-1}) = \frac{k}{p}, \qquad \operatorname{Var} T_k = \operatorname{Var}\sum_{i=1}^{k}(T_i - T_{i-1}) = \frac{kq}{p^2}.$$

This shows that stationary and independent increments is a powerful property of a process and makes the process characterization much simpler. Next, we show one additional property of this process.

Lemma 4.4. *The increments of time of success process $\{T_k : k \in \mathbb{N}\}$ are memoryless.*

Proof. It follows from the property of a geometric distribution, that for positive integers m, n

$$P\{T_{k+1} - T_k > m + n | T_{k+1} - T_k > m\} = \frac{P\{T_{k+1} - T_k > m + n\}}{P\{T_{k+1} - T_k > m\}} = q^n = P\{T_{k+1} - T_k > n\}.$$

□