Lecture 03: Poisson Process

1 Simple point processes

A **simple point process** is a collection of distinct points

$$\Phi = \{S_n \in \mathbb{R}^d : n \in \mathbb{N}\},$$

such that $|S_n| \to \infty$ as $n \to \infty$. In $\mathbb{R}_+$, one can order these points $\{S_n : n \in \mathbb{N}\}$. Let $N(\emptyset) = 0$ and denote the number of points in a set $A \subseteq \mathbb{R}^d$ by

$$N(A) = \sum_{n \in \mathbb{N}} 1\{S_n \in A\}.$$

Then $\{N(A) : A \in \mathscr{F}\}$ is called a **counting process** for the simple point process Φ. A counting process is **simple** if the jump size is unity.

1. Point processes can model arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Point processes can model location of nodes in a network, such as cellular networks, sensor networks, etc.

We can simplify this definition for $d = 1$. A stochastic process $\{N(t), t \geqslant 0\}$ is a **counting process** if

1. $N(0) = 0$, and
2. for each $\omega \in \Omega$, the map $t \mapsto N(t)$ is non-decreasing, integer valued, and right continuous.

Lemma 1.1. *A counting process has finitely many jumps in a finite interval* $[0,t)$.

The points of discontinuity correspond to the arrival instants of the point process $N(t)$. The n**th arrival instant** is a random variable denoted S_n, such that

$$S_0 = 0, \qquad S_n = \inf\{t \geq 0 : N(t) \geqslant n\},\ n \in \mathbb{N}.$$

The **inter arrival time** between $(n-1)$th and nth arrival is denoted by X_n and written as

$$X_n = S_n - S_{n-1}.$$

For a simple point process, we have

$$P\{X_n = 0\} = P\{X_n \leqslant 0\} = 0.$$

General point processes in higher dimension don't have any inter-arrival time interpretation.

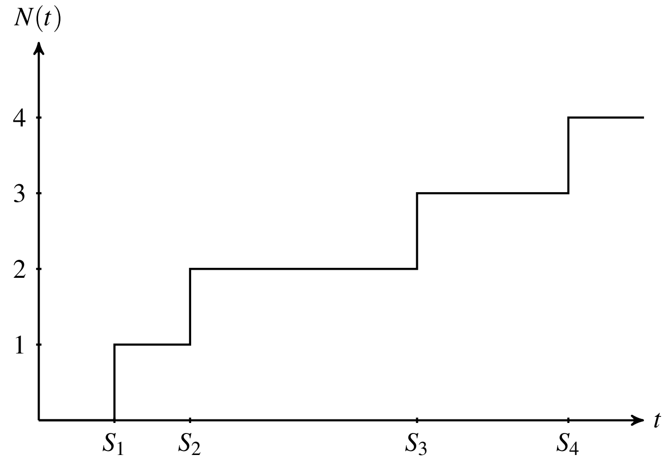

Figure 1: Sample path of a simple counting process.

Lemma 1.2. *Simple counting process* $\{N(t), t \geqslant 0\}$ *and arrival process* $\{S_n : n \in \mathbb{N}\}$ *are inverse processes. That is,*

$$\{S_n \leqslant t\} = \{N(t) \geqslant n\}.$$

Proof. Let $\omega \in \{S_n \leqslant t\}$, then $N(S_n) = n$ by definition. Since N is a non-decreasing process, we have $N(t) \geq N(S_n) = n$. Conversely, let $\omega \in \{N(t) \geqslant n\}$, then it follows from definition that $S_n \leq t$. □

Corollary 1.3. *The following identity is true.*

$$\{S_n \leqslant t, S_{n+1} > t\} = \{N(t) = n\}.$$

Proof. It is easy to see that

$$\{S_{n+1} > t\} = \{S_{n+1} \leqslant t\}^c = \{N(t) \geqslant n+1\}^c = \{N(t) < n+1\}.$$

Hence, the result follows by writing

$$\{N(t) = n\} = \{N(t) \geqslant n, N(t) < n+1\} = \{S_n \leqslant t, S_{n+1} > t\}.$$

□

Lemma 1.4. *Let* $F_n(x)$ *be the distribution function for* S_n*, then*

$$P_n(t) \triangleq P\{N(t) = n\} = F_n(t) - F_{n+1}(t).$$

Proof. It suffices to observe that following is a union of disjoint events,

$$\{S_n \leqslant t\} = \{S_n \leqslant t, S_{n+1} > t\} \cup \{S_n \leqslant t, S_{n+1} \leqslant t\}.$$

□

1.1 Stationary and independent increments

For an interval $I = (s,t]$, the number of arrivals in the interval I is defined as $N(I) = N(t) - N(s)$. Consider an arbitrary collection of mutually exclusive intervals $\{I_j : j \in [n]\}$, time index $t \geqslant 0$, and set of positive integers $\{k_j \in \mathbb{N}_0 : j \in [n]\}$. A counting process $\{N(t), t \geqslant 0\}$ has **stationary increments** if

$$P\{N(I_j) = k_j, j \in [n]\} = P\{N(t + I_j) = k_j, j \in [n]\}.$$

A counting process $\{N(t), t \geqslant 0\}$ has **independent increments** if

$$P\{N(I_j) = k_j, j \in [n]\} = \prod_{j \in [n]} P\{N(I_j) = k_j\}.$$

Lemma 1.5. *An arrival process $\{S_n, n \in \mathbb{N}_0\}$ has stationary and independent increments iff the sequence of inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables.*

Proof. We first suppose that $\{X_n : n \in \mathbb{N}\}$ is a sequence of *iid* random variables. Then $S_{n+m} - S_m$ has the same distribution as S_n and is independent of $(X_1, \dots, X_m)$. Conversely, we suppose that $\{S_n : n \in \mathbb{N}_0\}$ has stationary and independent increments. Then, $\{X_n : n \in \mathbb{N}\}$ is a sequence of *iid* random variables by looking at $X_n = S_n - S_{n-1}$. □

Lemma 1.6. *If a simple counting process $\{N(t), t \geqslant 0\}$ has stationary and independent increments then the sequence of inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are iid random variables.*

Proof. First, we notice that from inverse relationship, we have

$$\{X_n > y\} = \{N(S_{n-1}) \leqslant N(S_{n-1} + y) < N(S_n)\} = \{N(S_{n-1} + y) - N(S_{n-1}) = 0\}.$$

To show that each inter-arrival time is identically distributed, we utilize the stationarity of the increments of the counting process $N(t)$, to observe

$$P\{S_n - S_{n-1} > y\} = \int_0^\infty P\{N(y) = 0\} dF_{n-1}(t) = P\{N(y) = 0\} = P\{X_1 > y\}.$$

To show that inter-arrival times are independent, it suffices to show that X_n is independent of S_{n-1}. Since the increments of the counting process $N(t)$ are independent, we see that

$$\begin{aligned}
P\{S_{n-1} \leqslant x, X_n > y\} &= \int_0^x P\{N(y+t) - N(t) = 0 | S_{n-1} = t\} dF_{n-1}(t) \\
&= \int_0^x P\{N(y+t) - N(t) = 0 | N(t) = n-1, N(s) < n-1, s < t\} dF_{n-1}(t) \\
&= P\{X_n > y\} F_{n-1}(x).
\end{aligned}$$

□

2 Poisson process

A simple counting process $\{N(t),\ t \geqslant 0\}$ is called a **Poisson process** with a finite positive rate λ, if the inter-arrival times $\{X_n : n \in \mathbb{N}\}$ are *iid* random variables with an exponential distribution of rate λ. That is, it has a distribution function F, such that

$$F(x) = P\{X_1 \leqslant x\} = \begin{cases} 1 - e^{-\lambda x}, & x \geqslant 0 \\ 0, & \text{else.} \end{cases}$$

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N(t) = n\}$ for $n \in \mathbb{N}_0$. We need the following lemma that enables us to do that.

Lemma 2.1. *For any finite time $t > 0$, a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}$$

Fix $t > 0$ and we define a sample space subset $M = \{\omega \in \Omega : N(t)(\omega) = \infty\}$. For any $\omega \in M$, we have $S_n(\omega) \leqslant t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\}$. Hence, the probability measure for set M is zero. □

2.1 Memoryless distribution

A random variable X with continuous support on $\mathbb{R}_+$, is called **memoryless** if

$$P\{X > s\} = P\{X > t + s | X > t\} \ \forall t, s \in \mathbb{R}_+.$$

Proposition 2.2. *The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.*

Proof. Let X be a random variable with a memoryless distribution function $F : \mathbb{R}_+ \to [0,1]$. It follows that $\bar{F}(t) \triangleq 1 - F(t)$ satisfies the semi-group property

$$\bar{F}(t+s) = \bar{F}(t)\bar{F}(s).$$

Since $\bar{F}(x) = P\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $\bar{F}(x) = e^{\theta x}$, for some $\theta < 0$ from Lemma A.1. □

2.2 Distribution functions

Lemma 2.3. *Moment generating function of arrival times S_n is*

$$\mathbb{E}[e^{\theta S_n}] = \begin{cases} \frac{\lambda^n}{(\lambda-\theta)^n}, & \theta < \lambda \\ \infty, & \theta \geqslant \lambda. \end{cases}$$

Lemma 2.4. *Distribution function of S_n is given by*

$$F_n(t) \triangleq P\{S_n \leq t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}.$$

Theorem 2.5. *Density function of S_n is Gamma distributed with parameters n and λ. That is,*

$$f_n(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Theorem 2.6. *For each $t > 0$, the distribution of Poisson process $N(t)$ with parameter λ is given by*

$$P\{N(t) = n)\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Further, $\mathbb{E}[N(t)] = \lambda t$, explaining the rate parameter λ for Poisson process.

Proof. Result follows from density of S_n and recognizing that

$$P_n(t) = F_n(t) - F_{n+1}(t).$$

$\square$

Corollary 2.7. *Distribution of arrival times S_n is*

$$F_n(t) = \sum_{j \geq n} P_j(t), \qquad \sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E}N(t).$$

Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1\{N(t) \geqslant n\} = \sum_{n \in \mathbb{N}} P\{N(t) \geqslant n\} = \mathbb{E}N(t).$$

$\square$

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E}N(t) = \lambda t$, which is linearly increasing in time.

2.3 Age and excess time

At any time t, the instant of last and next arrivals are $S_{N(t)}$ and $S_{N(t)+1}$ respectively. **Age** of a counting process defined as age from the last arrival, and the **excess** is defined as remaining time till next arrival,

$$A(t) = t - S_{N(t)} \qquad Y(t) = S_{N(t)+1} - t.$$

Lemma 2.8. *Age and residual processes for a Poisson process are independent and the corresponding residual process has distribution same as inter-arrival distribution*

Proof. We first find the distribution of age $A(s)$ and excess time $Y(s)$ individually. Using stationary increment property of the counting process $N(t)$, we can write

$$P\{A(s) > x\} = \sum_{n \in \mathbb{N}_0} P\{N(s) - N(s-x) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(x) = 0, N(s-x) = n\} = P_0(x),$$
$$P\{Y(s) > y\} = \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y) = 0, N(s) = n\} = P_0(y).$$

Since the counting process $N(t)$ has stationary and independent increments, we can write the joint probability as

$$P\{A(s) > x, Y(s) > y\} = \sum_{n \in \mathbb{N}_0} P\{N(s+y) - N(s-x) = 0, N(s) = n\} = \sum_{n \in \mathbb{N}_0} P\{N(y+x) = 0, N(s-x) = n\}$$
$$= P\{N(y+x) = 0\} = P\{N(y+x) - N(y) = 0\} P\{N(y) = 0\} = P_0(x) P_0(y).$$

Therefore, $Y(s)$ is independent of $A(s)$ and they both have the same exponential distribution as X_{n+1}. The memoryless property of exponential distribution is crucially used. $\square$

A Functions with semigroup property

Lemma A.1. *A unique non-negative right continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying equation*

$$f(t+s) = f(t)f(s), \text{ for all } t,s \in \mathbb{R}$$

is $f(t) = e^{\theta t}$, *where* $\theta = \log f(1)$.

Proof. Clearly, we have $f(0) = f^2(0)$. Since f is non-negative, it means $f(0) = 1$. By definition of θ and induction for $m,n \in \mathbb{Z}^+$, we see that

$$f(m) = f(1)^m = e^{\theta m}, \qquad e^{\theta} = f(1) = f(1/n)^n.$$

Let $q \in \mathbb{Q}$, then it can be written as $m/n, n \neq 0$ for some $m,n \in \mathbb{Z}^+$. Hence, it is clear that for all $q \in \mathbb{Q}^+$, we have $f(q) = e^{\theta q}$. either unity or zero. Note, that f is a right continuous function and is non-negative. Now, we can show that f is exponential for any real positive t by taking a sequence of rational numbers $\{t_n\}$ decreasing to t. From right continuity of f, we obtain

$$f(t) = \lim_{t_n \downarrow t} f(t_n) = \lim_{t_n \downarrow t} e^{\beta t_n} = e^{\beta t}.$$

□