Lecture 04: Properties of Poisson Process

1 Characterizations of Poisson process

It is clear that t partitions $X_{N(t)+1}$ in two parts such that $X_{N(t)+1} = A(t) + Y(t)$ as seen in Figure **??** for the case when $N(s) = n$.

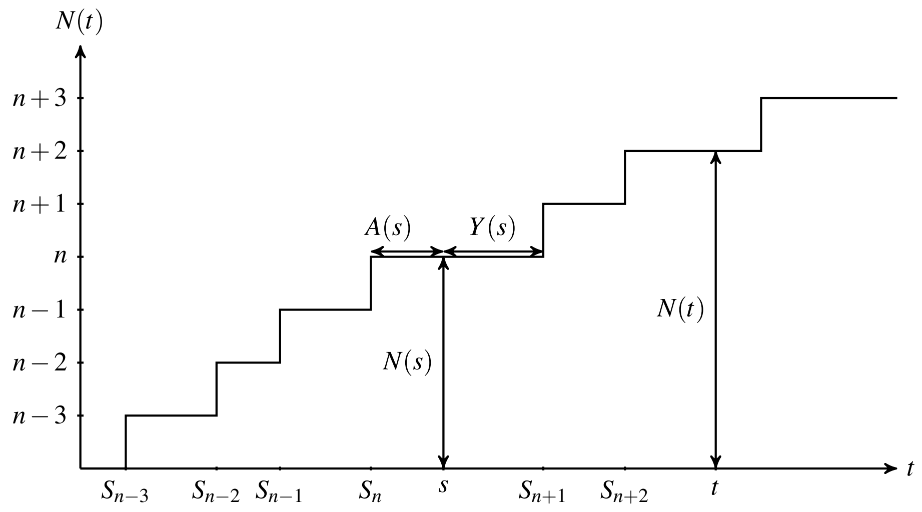

Figure 1: Stationary and independent increment property of Poisson process.

Proposition 1.1. *A Poisson process* $\{N(t), t \geqslant 0\}$ *is simple counting process with stationary independent increments.*

Proof. It is clear that Poisson process is a simple counting process. To show that $N(t)$ has stationary and independent increments, it suffices to show that $N(t) - N(s)$ is independent of $N(s)$ and the distribution of increment $N(t) - N(s)$ is identical to that of $N(t-s)$. This follows from the fact that we can use induction to show stationary and independent increment property for for any finite disjoint time-intervals.

We can write the distribution of $N(t) - N(s)$ given $N(s)$ in terms of the following events involving inter-arrival times and excess times as

$$P\{N(t) - N(s) \geqslant m | N(s) = n\} = P\{Y(s) + S_{n+m} - S_{n+1} \leqslant t - s | S_n + A(s) = s\}.$$

Further, we see that independent increment holds only if inter-arrival time is exponential. Since, $\{X_i : i \geqslant n+2\} \cup \{Y(s)\}$ are independent of $\{X_i : i \leqslant n\} \cup A(s)$, we have $N(t) - N(s)$ independent of $N(s)$. Further, since $Y(s)$ has same distribution as X_{n+1}, we get $N(t) - N(s)$ having same distribution as $N(t-s)$. By induction, we can extend this result to $(N(t_n) - N(t_{n-1}), \ldots, N(s))$. □

Theorem 1.2 (Characterization 1). *A simple counting process with stationary and independent increment is a Poisson process with parameter λ when*

$$\lim_{t\downarrow 0}\frac{P\{N(t)=1\}}{t}=\lambda, \qquad \lim_{t\downarrow 0}\frac{P\{N(t)\geq 2\}}{t}=0.$$

Proof. It suffices to show that first inter-arrival times X_1 is exponentially distributed with parameter λ. Notice that, the probability $P_0(t)$ of no arrivals in a time duration $[0,t)$ satisfies the semi-group property. That is,

$$P_0(t+s)=P\{N(t+s)-N(t)=0,N(t)=0\}=P_0(t)P_0(s).$$

Using the conditions in the theorem, the result follows. □

Proposition 1.3 (Characterization 2). *Let $\{I_i\subseteq\mathbb{R}_+ : i\in[k]\}$ be a finite collection of disjoint intervals. A stationary and independent increment simple counting process $\{N(t),\ t\geqslant 0\}$ with $N(0)=0$ is Poisson process iff*

$$P\bigcap_{i=1}^{k}\{N(I_i)=n_i\}=\prod_{i=1}^{k}\frac{(\lambda|I_i|)^{n_i}}{n_i!}e^{-\lambda|I_i|}.$$

Proof. It is clear that Poisson process satisfies the above conditions. Further, since $P\{N(t)=0\}=e^{-\lambda t}$, it follows that the counting process with stationary and independent increment is Poisson with rate λ. □

Proposition 1.4. *Let $\{N(t),t\geqslant 0\}$ be a Poisson process with $\{I_i\subseteq\mathbb{R}_+ : i\in[n]\}$ a set of finite disjoint intervals with $I=\cup_{i\in[n]}I_i$, and $\{k_i\in\mathbb{N}_0 : i\in[n]\}$ and $k=\sum_{i\in[n]}k_i$. Then, we have*

$$P\{N(I_i)=k_i,i\in[n]|N(I)=k\}=k!\prod_{i\in[n]}\frac{1}{k_i!}\left(\frac{|I_i|}{|I|}\right)^{k_i}.$$

Proof. It follows from the stationary and independent increment property of Poisson processes that

$$P\{N(I_i)=k_i,i\in[n]|N(I)=k\}=\frac{P\bigcap_{i\in[n]}\{N(I_i)=k_i\}}{P\{N(I)=k\}}=\frac{\prod_{i\in[n]}P\{N(I_i)=k_i\}}{P\{N(I)=k\}}.$$

□

1.1 Conditional distribution of arrivals

Proposition 1.5. *For a Poisson process $\{N(t),t\geqslant 0\}$, distribution of first arrival instant S_1 conditioned on $\{N(t)=1\}$ is uniform between $[0,t)$.*

Proof. If $N(t)=1$, then we know that conditional distribution of S_1 is supported on $[0,t)$. By Proposition **??**, we see that

$$P\{S_1\leq s|N(t)=1\}=P\{N(s)=1,N(t-s)=0|N(t)=1\}1\{s<t\}+1\{s\geq t\}=\frac{s}{t}1\{s<t\}+1\{s\geq t\}.$$

□

Proposition 1.6. *For a Poisson process $\{N(t),t\geqslant 0\}$, joint distribution of arrival instant $\{S_1,\dots,S_n\}$ conditioned on $\{N(t)=n\}$ is identical to joint distribution of order statistics of n iid uniformly distributed random variables between $[0,t]$.*

Proof. Let $\{s_i \in (0,t) : i \in [n]\}$ be a sequence of increasing numbers. If we denote $s_0 = 0$, then we can write

$$\bigcap_{i=1}^{n}\{S_i = s_i\} \cap \{N(t) = n\} \iff \bigcap_{i=1}^{n}\{X_i = s_i - s_{i-1}\} \cap \{X_{n+1} > t - s_n\}.$$

Note that all the events on RHS are independent. Hence, it is easy to compute the joint distribution of $\{S_1,\dots,S_n\}$ as

$$\begin{aligned} P\bigcap_{i=1}^{n}\{S_i \leq s_i\} \cap \{N(t) = n\} &= \int_0^{s_1} du_1 \cdots \int_0^{s_n} du_n \prod_{i=1}^{n} \lambda \exp(-\lambda(u_i - u_{i-1}) \exp(-\lambda(t - u_n)) \\ &= \lambda^n \exp(-\lambda t) \prod_{i=1}^{n} s_i. \end{aligned}$$

Since $P\{N(t) = n\} = \exp(-\lambda t)(\lambda t)^n/n!$, it follows that

$$P\{S_1 \leq s_1,\dots,S_n \leq s_n | N(t) = n\} = \begin{cases} n!\prod_{i=1}^{n} \frac{s_i}{t} & s < t \\ 0 & s \geq t. \end{cases}$$

Let $U_1,\dots,U_n$ be *iid* uniform random variables in $[0,t]$. Then, the order statistics of $U_1 \dots,U_n$ has an identical joint distribution to n arrival instants conditioned on $\{N(t) = n\}$. □

2 Superposition and decomposition of Poisson processes

Theorem 2.1 (Sum of Independent Poissons). *Let $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ be two independent Poisson processes with rats λ_1 and λ_2 respectively. Then, the process $N(t) = N_1(t) + N_2(t)$ is Poisson with rate $\lambda_1 + \lambda_2$.*

Proof. We need to show that $\{N(t)\}$ has stationary independent increments, and

$$P\{N(t) = n\} = \exp(-(\lambda_1 + \lambda_2)t)\frac{(\lambda_1 + \lambda_2)^n t^n}{n!}.$$

For two disjoint interval (t_1,t_2) and (t_3,t_4), we can see that for both processes $N_1(t)$ and $N_2(t)$, arrivals in (t_1,t_2) and (t_3,t_4) are independent. Therefore, $N(t)$ has independent increment property. Similarly, we can argue about the stationary increment property of $\{N(t)\}$. Further, we can write

$$\{N(t) = n\} = \bigcup_{k=0}^{n}\{\{N_1(t) = k\} \cap \{N_2(t) = n - k\}\}.$$

Since $N_1(t)$ and $N_2(t)$ are independent, we can write

$$\begin{aligned} P\{N(t) = n\} &= \sum_{k=0}^{n} \exp(-\lambda_1 t)\frac{(\lambda_1 t)^k}{k!} \exp(-\lambda_2 t)\frac{(\lambda_2 t)^{n-k}}{(n-k)!}, \\ &= \frac{\exp(-(\lambda_1 + \lambda_2)t)}{n!} \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 t)^k (\lambda_2 t)^{n-k}. \end{aligned}$$

Result follows by recognizing that summand is just binomial expansion of $[(\lambda_1 + \lambda_2)t]^n$. □

Remark 2.2. If independence condition is removed, the statement is not true.

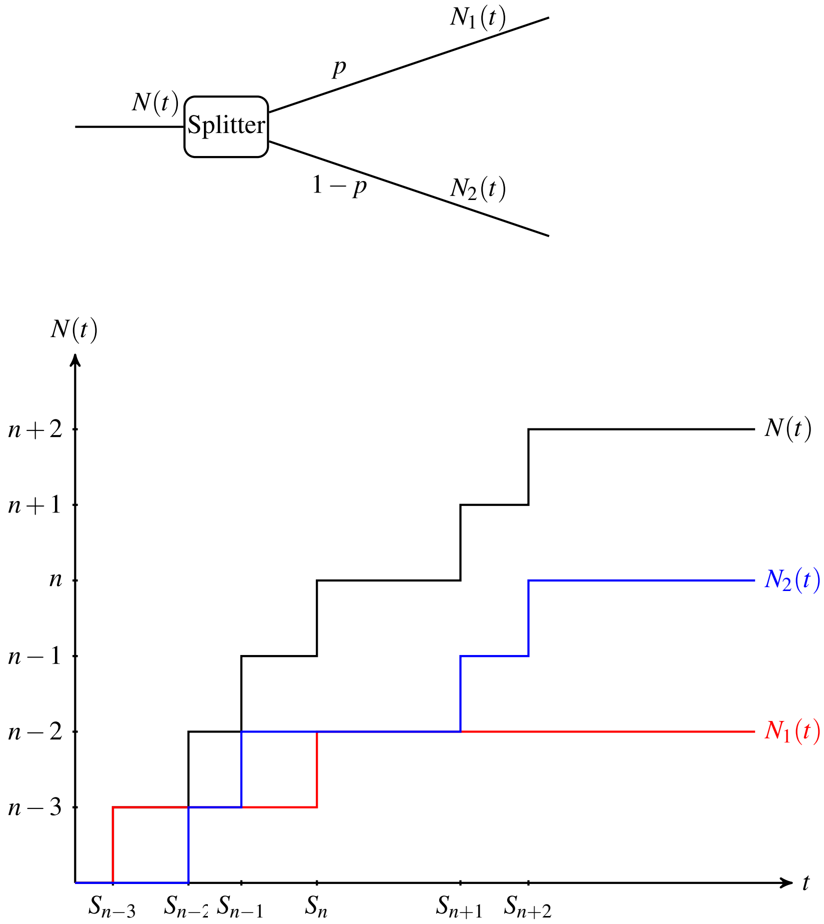

Figure 2: Splitting a Poisson process into two independent Poisson processes.

Theorem 2.3 (Independent Spilitting). *Let $\{N(t), t \geqslant 0\}$ be a Poisson arrival process. Each arrival can be randomly assigned to either arrival type 1 or 2, with probability p and $(1-p)$ respectively, independent of previous assignments. Arrival processes of type 1 and 2 are denoted by $N_1(t)$ and $N_2(t)$ respectively. Then, $\{N_1(t), t \geqslant 0\}$,and $\{N_2(t), t \geqslant 0\}$ are mutually independent Poisson processes with rates λp and $\lambda(1-p)$ respectively.*

Proof. To show that $N_1(t), t \geq 0$ is a Poisson process with rate λp, we show that it is stationary independent increment process with the distribution

$$P\{N_1(t) = n\} = \frac{(p\lambda t)^n}{n!} e^{-\lambda p t}.$$

The stationary, independent increment property of the probabilistically filtered processes $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ can be understood and argued out from the example given in the figure. Notice that

$$\{N_1(t) = k\} = \bigcup_{n=k}^{\infty} \{N(t) = n, N_1(t) = k\}.$$

Further notice that conditioned on $\{N(t) = n\}$, probability of event $\{N_1(t) = k\}$ is merely probability of selecting k arrivals out of n, each with independent probability p. Therefore,

$$P\{N_1(t) = k\} = \exp(-\lambda t) \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} \binom{n}{k} p^k (1-p)^{n-k},$$
$$= \exp(-\lambda t) \frac{(\lambda p t)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p)t)^{n-k}}{(n-k)!}.$$

Recognizing that infinite sum in RHS adds up $\exp(\lambda(1-p)t)$, the result follows. We can find the distribution of $N_2(t)$ by similar arguments. We will show that events $\{N_1(t) = n_1\}$ and $\{N_2(t) = n_2\}$ are independent. To this end, we see that

$$\{N_1(t) = n_1, N_2(t) = n_2\} = \{N(t) = n_1 + n_2, N_1(t) = n_1\}.$$

Using their distribution for $N_1(t), N_2(t)$, and conditional distribution of $N_1(t)$ on $N(t)$, we can show that

$$P\{N_1(t) = n_1, N_2(t) = n_2\} = \exp(-\lambda t) \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \binom{n_1+n_2}{n_1} p^{n_1}(1-p)^{n_2},$$
$$= P\{N_1(t) = n_1\} P\{N_2(t) = n_2\}.$$

In general, we need to show finite dimensional distributions factorize. That is, we need to show that for measurable sets $A_1, \dots, A_n : j \in [m]\}$, we have

$$P\left(\bigcap_{i=1}^{n}\{N_1(t_i) \in A_i\} \bigcap_{j=1}^{m}\{N_2(s_j) \in B_j\}\right) = P\left(\bigcap_{i=1}^{n}\{N_1(t_i) \in A_i\}\right) P\left(\bigcap_{j=1}^{m}\{N_2(s_j) \in B_j\}\right).$$

□

A Order statistics

For any n length sequence $a \in \mathbb{R}^n$, the **order statistics** is a permutation $\sigma : [n] \to [n]$ such that

$$a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(n)}.$$

For, $k \in [n]$, we call $a_{\sigma(k)}$ as the k**th order statistic** of the sequence a. In particular, first order statistic is the minimum, and the nth order statistic is the maximum of a n length sequence.

Lemma A.1. *Let $X = (X_1, X_2, \dots, X_n)$ be an n-length sequence of iid random variables with common distribution and density functions F and f respectively. Then, the joint density of order statistics of sequence X for $x \in \mathbb{R}^n$ is*

$$f_{X \circ \sigma}(x) = n! \prod_{i=1}^{n} f(x_i).$$

Lemma A.2. *Let $X = (X_1, X_2, \dots, X_n)$ be an n-length sequence of iid random variables with common distribution and density functions F and f respectively. Then, the density function of kth order statistic of sequence X for $x \in \mathbb{R}$ is*

$$f_{X_{\sigma(k)}}(x) = \binom{n}{k} F(x)^{k-1} \bar{F}(x)^{n-k} f(x).$$