Lecture 05: Compound and Non-Stationary Poisson Processes

1 Compound Poisson process

A **compound Poisson process** is a real-valued point process $\{Z_t, t \geq 0\}$ having the following properties.

1. finite jumps: for all $\omega \in \Omega, t \longmapsto Z_t(\omega)$ has finitely many jumps in finite intervals.
2. independent increments: for all $t, s \geq 0; Z_{t+s} - Z_t$ is independent of past $\{Z_u, u \leq t\}$.
3. stationary increments: for all $t, s \geq 0$, distribution of $Z_{t+s} - Z_t$ depends only on s and not on t.

For each $\omega \in \Omega$, we can define time and size of nth jump

$$S_0(\omega) = 0 \qquad S_n(\omega) = \inf\{t > 0 : Z_t(\omega) > Z_{S_{n-1}}(\omega)\},\ n \in \mathbb{N},$$
$$X_0(\omega) = 0 \qquad X_n(\omega) = Z_n(\omega) - Z_{n-1}(\omega).$$

Let $N_t, t \geqslant 0$ be the counting process associated with the number of jumps in $[0,t)$. Then, S_n are the arrival instants of nth jumps.

Proposition 1.1. *A stochastic process $\{Z_t, t \geqslant 0\}$ is a compound Poisson process iff its jump times form a Poisson process and the jump sizes form an iid random sequence independent of the jump times.*

Proof. From independent increment property of compound Poisson processes, it follows that $Z_{t+s} - Z_t = 0$ is independent of the past $Z_u, u \leqslant t$. Further, it follows from the stationary increment property that the distribution of $Z_{t+s} - Z_t = 0$ is independent of t. It follows that N_t is a Poisson process. Similarly, it follows that $X_1, X_2, \dots$ are *iid* random variables, independent of $S_1, S_2, \dots$.

Conversely, let $Z_t = \sum_{i=1}^{N_t} X_i$ where N_t is a Poisson process independent of the random *iid* sequence $X_1, X_2, \dots$. It is easy to check that Z_t has finitely many jumps in finite intervals. Further, one can show independent and stationary increment properties. □

- Arrival of customers in a store is a Poison process N_t. Each customer i spends an *iid* amount X_i independent of the arrival process. Amount of money spent Y_n by first n customers is

$$Y_0 = 0, \qquad Y_n = \sum_{i=1}^{n} X_i, i \in [n].$$

  Now define $Z_t = Y_{N_t}$ as the amount spent by the customers arriving in time t. Then $\{Z_t, t \geq 0\}$ is a compound Poisson Process.

- Let the time between successive failures of a machine be independent and exponentially distributed. The cost of repair is *iid* random at each failure. Then the total cost of repair in a certain time t is a compound Poisson Process.

2 Non-stationary Poisson process

From the characterization of Poisson process just stated, we can generalize to non-homogeneous Poisson Process. In this case, the rate of Poisson Process λ is time varying.

An integer valued counting process $\{N(t),\ t \geqslant 0\}$ is said to be possibly **non-stationary Poisson process** if it has unit jumps and independent increments. That is,

1. for each $\omega \in \Omega$, the map $t \mapsto N_t(\omega)$ has jumps of unit size only,
2. for any $t, s \geq 0$, the random variable $N_{t+s} - N_t$ is independent of the past $\{N_u, u \leqslant t\}$.

Let $m(t) = \mathbb{E}N_t$ for all $t \geqslant 0$. From non-decreasing property of counting processes, it follows that the mean is also non-decreasing in time t. From right continuity of counting process and the monotone convergence theorem, it follows that mean function is also right continuous. The **time inverse** of mean is defined as

$$\tau(t) = \inf\{s > 0 : m(s) > t\},\ t \geqslant 0.$$

Since, inverse of a non-decreasing function is also non-decreasing, we conclude that $\tau(t)$ is non-decreasing function of time t.

Theorem 2.1. *Let N_t be a non-stationary Poisson process, such that $m(t) = \mathbb{E}N_t$ is continuous. Then,*

$$M_t(\omega) \triangleq N_{\tau(t)}(\omega),\ \ t \geqslant 0, \omega \in \Omega,$$

is a stationary Poisson process with unit rate.

Proof. Fix $t > s \geq 0$ and let $s' \triangleq \tau(s)$ and $t' \triangleq \tau(t) - \tau(s)$. Then, by definition of M_t, t', s' and independent increment property of non-stationary Poisson process N_t, we have

$$\mathbb{E}[M_t - M_s | M_u; u \leqslant s] = \mathbb{E}[N_{t'} - N_{s'} | N_u; u \leqslant s'] = m(t') - m(s') = m(\tau(t)) - m(\tau(s)) = t - s.$$

It follows that M_t is a simple counting process with independent and stationary increments and unit rate. □

Corollary 2.2. *Let $m(t)$ be a continuous non-decreasing function. Then, $S_1, S_2, \dots$ are the arrival instants in a non-stationary Poisson process N_t with mean function $m(t) = \mathbb{E}N_t$ iff $m(S_1), m(S_2), \dots$ are the arrivals instants of a stationary Poisson process of unit rate.*

Proof. We can write the nth arrival instant S'_n of unit-rate stationary Poisson process M_t, in terms of the nth arrival instant S_n of non-stationary Poisson process N_t as

$$S'_n = \inf\{t > 0 : \tau(t) > S_n\} = \inf\{t > 0 : m(S_n) > t\} = m(S_n).$$

□

This corollary implies that $S_n \in [s,t)$ if and only if $m(S_n) \in [m(s), m(t))$. Therefore, number of arrivals in $[s,t)$ equals number of arrivals for unit-rate stationary Poisson process in $[m(s), m(t))$. Hence, we conclude that for $b(s,t) = m(t) - m(s)$

$$\Pr\{N_t - N_s = k\} = e^{-b(s,t)} \frac{b(s,t)^k}{k!},\ k \in \mathbb{N}_0.$$

We will see that the inter-arrival times for the non-stationary Poisson process N_t, defined as

$$T_0 = 0, \qquad\qquad T_n = S_n - S_{n-1},\ n \in \mathbb{N},$$

are not independent anymore.

Proposition 2.3. *For a non-stationary Poisson process with continuous mean function $m(t)$, we have*

$$\Pr\{T_{n+1} > t | S_1, S_2, \dots, S_n\} = \exp\left(-m(S_n+t) + m(S_n)\right).$$

Proof. We define events $A = \{m(S_{n+1}) > m(S_n+t)\}$ and $B = \{m(S_{n+1}) \geq m(S_n+t)\}$. Then, we have $A \subseteq \{T_{n+1} > t\} \subseteq B$. Hence, we can write

$$\Pr\{A | S_1, S_2, \dots, S_n\} \leq \Pr\{T_{n+1} > t | S_1, S_2, \dots, S_n\}.$$

The arrival instants $S_1, \dots, S_n$ determine $m(S_1), \dots, m(S_n), m(S_n+t)$. Further, since $m(S_{n+1}) - m(S_n)$ is the inter-arrival time of the stationary Poisson process $M(t)$, it is independent of $S_1, S_2, \dots, S_n$

$$\Pr\{A | S_1, \dots, S_n\} = \Pr\{m(S_{n+1}) - m(S_n) > m(S_n+t) - m(S_n) | S_1, \dots, S_n\} = \exp\left(-m(S_n+t) + m(S_n)\right).$$

Result follows from the continuity of the exponential distribution. □

A Laplace functional

The **Laplace functional** $\mathcal{L}$ of a point process Φ and associated counting process N is defined for all non-negative function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathcal{L}_\Phi(f) = \mathbb{E}\exp\left(-\int_{\mathbb{R}^d} f(x)dN(x)\right).$$

For simple function $f(x) = \sum_{i=1}^k t_i 1\{x \in A_i\}$, we can write the Laplace functional

$$\mathcal{L}_\Phi(f) = \mathbb{E}\exp(-\sum_i t_i N(A_i)),$$

as a function of the vector $(t_1, t_2, \dots, t_k)$, a joint Laplace transform of the random vector $(N(A_1), \dots, N(A_k))$. This way, one can compute all finite dimensional distribution of the counting process N.

Proposition A.1. *The Laplace functional of the Poisson process with intensity measure Λ is*

$$\mathcal{L}_\Phi(f) = \exp\left(-\int_{\mathbb{R}^d} (1 - e^{-f(x)})\Lambda(dx)\right).$$

Proof. For a bounded Borel measurable set $A \subseteq \mathbb{R}^d$, consider $g(x) = f(x)1\{x \in A\}$. Then,

$$\mathcal{L}_\Phi(g) = \mathbb{E}\exp(-\int_{\mathbb{R}^d} g(x)dN(x)) = \mathbb{E}\exp(-\int_A f(x)dN(x)).$$

Clearly $dN(x) = \delta_x 1\{x \in \Phi\}$ and hence we can write

$$\mathcal{L}_\Phi(g) = \mathbb{E}\exp\left(-\sum_{S_i \in \Phi \cap A} \int_A f(S_i)\right).$$

We know that the probability of $N(A) = |\Phi(A)| = n$ points in set A is given by

$$P\{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}.$$

Given there are n points in set A, the density of n point locations are independent and given by

$$f_{S_1,\dots,S_n}(x_1,\dots,x_n) = \left(\frac{\Lambda(dx)}{\Lambda(A)}\right)^n, \; x_1,\dots,x_n \in A.$$

Hence, we can write the Laplace functional as

$$\mathcal{L}_\Phi(g) = e^{-\Lambda(A)} \sum_{n\in\mathbb{N}_0} \frac{\Lambda(A)^n}{n!} \prod_{i=1}^{n} \int_A e^{-f(x_i)} \frac{\Lambda(dx_i)}{\Lambda(A)} = \exp\left(-\int_{\mathbb{R}^d} (1-e^{-g(x)})\Lambda(dx)\right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. □

A.1 Superposition of point processes

Theorem A.2. *The superposition of independent Poisson point processes with intensities Λ_k is a Poisson point process with intensity measure $\sum_k \Lambda_k$ if and only if the latter is a locally finite measure.*

A.2 Thinning of point processes

Consider a probability **retention function** $p : \mathbb{R}^d \to [0,1]$ and a point process Φ. The **thinning** of point process $\Phi = \{S_n \in \mathbb{R}^d : n \in \mathbb{N}\}$ with the retention function p is a point process such that

$$\Phi^p = \{S_n \in \Phi : Y(S_n) = 1\},$$

where $Y(S_n)$ is an independent indicator stochastic process at each point S_n and $\mathbb{E}Y(S_n) = p(S_n)$.

Theorem A.3. *The thinning of the Poisson point process of intensity measure Λ with the retention probability function p yields a Poisson point process of intensity measure $p\Lambda$ with*

$$(p\Lambda)(A) = \int_A p(x)\Lambda(dx)$$

for all bounded Borel measurable $A \subseteq \mathbb{R}^d$.

Proof. Let $A \subseteq \mathbb{R}^d$ be a bounded Boreal measurable set, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function. Consider the Laplace functional of the thinned point process Φ^p for a non-negative function $g(x) = f(x)1\{x \in A\}$

$$\mathcal{L}_{\Phi^p}(g) = e^{-\Lambda(A)} \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \left(\int_A (p(x)e^{-f(x)} + (1-p(x))\Lambda(dx)\right)^n = \exp\left(-\int_{\mathbb{R}^d} (1-e^{-g(x)})p(x)\Lambda(dx)\right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. □