Lecture 06: Renewal Theory

1 Introduction

One of the characterization for the Poisson process is of it being a counting process with *iid* exponential inter-arrival times. Now we shall relax the "exponential" part. As a result, we no longer have the nice properties such as independent and stationary increments that Poisson processes had. However, we can still get some great results which also apply to Poisson Processes.

1.1 Renewal instants

We will consider **inter-arrival times** $\{X_i : i \in \mathbb{N}\}$ to be a sequence of non-negative *iid* random variables with a common distribution F, with finite mean μ and $F(0) < 1$. We interpret X_n as the time between $(n-1)^{\text{st}}$ and the n^{th} renewal event. Let S_n denote the time of n^{th} **renewal instant** and assume $S_0 = 0$. Then, we have

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.$$

Second condition on inter-arrival times implies non-degenerate renewal process. If $F(0)$ is equal to 1 then it is a trivial process. A counting process $\{N(t), t \geq 0\}$ with *iid* general inter-arrival times is called a **renewal process**, written as

$$N(t) = \sup\{n \in \mathbb{N}_0 : S_n \leq t\} = \sum_{n \in \mathbb{N}} 1_{\{S_n \leq t\}}.$$

Lemma 1.1 (Inverse Relationship). *There is an inverse relationship between time of n^{th} event S_n, and the counting process $N(t)$. That is*

$$\{S_n \leq t\} \iff \{N(t) \geq n\}. \tag{1}$$

Lemma 1.2 (Finiteness of $N(t)$). *For all $t > 0$, the number of renewals $N(t)$ in time $[0,t)$ is finite.*

Proof. We are interested in knowing how many renewals occur per unit time. From strong law of large numbers, we know that the set

$$\left\{ \frac{S_n}{n} = \mu, n \in \mathbb{N} \right\},$$

has probability measure unity. Further, since $\mu > 0$, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Indeed, the following set

$$\{N(t) \geq n, n \in \mathbb{N}\} = \{S_n \leq t, n \in \mathbb{N}\} = \left\{ \frac{S_n}{n} \leq \frac{t}{n}, n \in \mathbb{N} \right\}.$$

has measure zero for any finite t. Therefore, $N(t)$ must be finite, and $N(t) = \max\{n \in \mathbb{N}_0 : S_n \leq t\}$. $\square$

1.2 Distribution functions

The distribution of renewal instant S_n is denoted by $F_n(t) \triangleq \Pr\{S_n \le t\}$ for all $t \in \mathbb{R}$.

Lemma 1.3. *The distribution function F_n for renewal instant S_n can be computed inductively*

$$F_1 = F, \qquad F_n = F_{n-1} * F \triangleq \int_0^t F_{n-1}(t-y)dF(y),$$

where $*$ denotes convolution.

Proof. It follows from induction over sum of *iid* random variables. □

Lemma 1.4. *Counting process $N(t)$ assumes non-negative integer values with distribution*

$$\Pr\{N(t) = n\} = \Pr\{S_n \le t\} - \Pr\{S_{n+1} \le t\} = F_n(t) - F_{n+1}(t).$$

Proof. It follows from the inverse relationship between renewal instants and the renewal process (1). □

Mean of the counting process $N(t)$ is called the **renewal function** denoted by $m(t) = \mathbb{E}[N(t)]$.

Proposition 1.5. *Renewal function can be expressed in terms of distribution of renewal instants as*

$$m(t) = \sum_{n \in \mathbb{N}} F_n(t).$$

Proof. Using the inverse relationship between counting process and the arrival instants, we can write

$$m(t) = \mathbb{E}[N(t)] = \sum_{n \in \mathbb{N}} \Pr\{N(t) \ge n\} = \sum_{n \in \mathbb{N}} \Pr\{S_n \le t\} = \sum_{n \in \mathbb{N}} F_n(t).$$

We can exchange integrals and summations since the integrand is positive using monotone convergence theorem. □

Proposition 1.6. *Renewal function is bounded for all finite times.*

Proof. Since we assumed that $\Pr\{X_n = 0\} < 1$, it follow from continuity of probabilities that there exists $\alpha > 0$, such that $\Pr\{X_n \ge \alpha\} = \beta > 0$. Define

$$\bar{X}_n = \alpha 1_{\{X_n \ge \alpha\}}.$$

Note that since X_i's are *iid*, so are $\bar{X}_i$'s, which are bivariate random variables taking values in $\{0, \alpha\}$ with probabilities $1-\beta$ and β respectively. Let $\bar{N}(t)$ denote the renewal process with inter-arrival times $\bar{X}_n$, with arrivals at integer multiples of α. Since $\bar{X}_n \le X_n$, we have $\bar{N}(t) \ge N(t)$ for all sample paths. Hence, it follows that $\mathbb{E}N(t) \le \mathbb{E}\bar{N}(t)$, and we will show that $\mathbb{E}\bar{N}(t)$ is finite. We can write the joint distribution of number of arrivals at each arrival instant $l\alpha$, as

$$\begin{aligned}\Pr\{\bar{N}(0) = n_1, \bar{N}(\alpha) = n_2\} &= \Pr\{X_i \le \alpha, i \le n_1, X_{n_1+1} \ge \alpha, X_i < \alpha, n_1 + 2 \le i \le n_2, X_{n_2+1} \ge \alpha\} \\ &= (1-\beta)^{n_1}\beta(1-\beta)^{n_2-1}\beta\end{aligned}$$

It follows that the number of arrivals is independent at each arrival instant $k\alpha$ and geometrically distributed with mean $1/\beta$ and $(1-\beta)/\beta$ for $k \ge 1$ and $k = 0$ respectively. Thus, for all $t \ge 0$,

$$\mathbb{E}N(t) \le \mathbb{E}[\bar{N}(t)] \le \frac{\lceil \frac{t}{\alpha} \rceil}{\beta} \le \frac{\frac{t}{\alpha} + 1}{\beta} < \infty.$$

□

1.3 Basic renewal theorem

Lemma 1.7. *Let* $N(\infty) \triangleq \lim_{t\to\infty} N(t)$. *Then,* $\Pr\{N(\infty) = \infty\} = 1$.

Proof. It suffices to show $\Pr\{N(\infty) < \infty\} = 0$. Since $\mathbb{E}[X_n] < \infty$, we have $\Pr\{X_n = \infty\} = 0$ and

$$\Pr\{N(\infty) < \infty\} = \Pr \bigcup_{n\in\mathbb{N}} \{N(\infty) < n\} = \Pr \bigcup_{n\in\mathbb{N}} \{S_n = \infty\} = \Pr\{\bigcup_{n\in\mathbb{N}} \{X_n = \infty\}\} \le \sum_{n\in\mathbb{N}} \Pr\{X_n = \infty\} = 0.$$

$\square$

Notice that $N(t)$ increases to infinity with time. We are interested in rate of increase of $N(t)$ with t.

Theorem 1.8 (Basic Renewal Theorem).

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \textit{almost surely}.$$

Proof. Note that $S_{N(t)}$ represents the time of last renewal before t, and $S_{N(t)+1}$ represents the time of first renewal after time t. Consider $S_{N(t)}$. By definition, we have

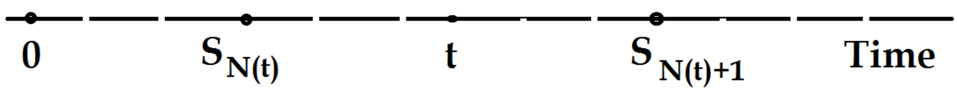

Figure 1: Time-line visualization

$$S_{N(t)} \le t < S_{N(t)+1}$$

Dividing by $N(t)$, we get

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

By Strong Law of Large Numbers (SLLN) and the previous result, we have

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu \quad \text{a.s.}$$

Also

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

Hence by squeeze theorem, the result follows. $\square$

Suppose, you are in a casino with infinitely many games. Every game has a probability of win X, *iid* uniformly distributed between $(0,1)$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins. Let $N(n)$ denote the number of losses in n plays. Then the fraction of wins $P_W(n)$ is given by

$$P_W(n) = \frac{n - N(n)}{n}.$$

We pick a strategy where any game is selected to play, and continue to be played till the first loss. Note that, time till first loss is geometrically distributed with mean $\frac{1}{1-X}$. We shall show that this fraction approaches unity as $n \to \infty$. By the previous proposition, we have:

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}[\text{Time till first loss}]}$$
$$= \frac{1}{\mathbb{E}\left[\frac{1}{1-X}\right]} = \frac{1}{\infty} = 0$$

Hence Renewal theorems can be used to compute these long term averages. We'll have many such theorems in the following sections.

1.4 Elementary renewal theorem

Basic renewal theorem implies $N(t)/t$ converges to $1/\mu$ almost surely. Now, we are interested in convergence of $\mathbb{E}[N(t)]/t$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean.

Consider the following example. Let X_n be a Bernoulli random variable with $\Pr\{X_n = 1\} = 1/n$. Let $Y_n = nX_n$. Then, $\Pr\{Y_n = 0\} = 1 - 1/n$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. So $\mathbb{E}[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $\mathbb{E}[N(t)]/t$ converging to $1/\mu$.

Theorem 1.9 (Elementary renewal theorem). *Let $m(t)$ denote mean $\mathbb{E}[N(t)]$ of renewal process $N(t)$, then under the hypotheses of basic renewal theorem, we have*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Proof. Take $\mu < \infty$. We know that $S_{N(t)+1} > t$. Therefore, taking expectations on both sides and using Proposition A.2, we have

$$\mu(m(t) + 1) > t.$$

Dividing both sides by μt and taking $\liminf$ on both sides, we get

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{X}_n = X_n 1_{\{X_n \leq M\}} + M 1_{\{X_n > M\}}.$$

We will call $\mathbb{E}[\bar{X}_n] = \mu_M$. Further, we can define arrival instants $\{\bar{S}_n\}$ and renewal process $\bar{N}(t)$ for this set of truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{S}_n = \sum_{k=1}^{n} \bar{X}_k, \qquad \bar{N}(t) = \sup\{n \in \mathbb{N}_0 : \bar{S}_n \leq t\}.$$

Note that since $S_n \geq \bar{S}_n$, the number of arrivals would be higher for renewal process $\bar{N}(t)$ with truncated random variables, i.e.

$$N(t) \leq \bar{N}(t). \tag{2}$$

Further, due to truncation of inter-arrival time, next renewal happens with-in M units of time, i.e.

$$\bar{S}_{\bar{N}(t)+1} \leq t + M.$$

Taking expectations on both sides in the above equation, using Wald's lemma for renewal processes, dividing both sides by $t\mu_M$, and taking lim sup on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}.$$

Taking expectations on both sides of (2) and letting M go arbitrary large on RHS, we get

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

Result follows for finite μ from combining liminf and limsup of the $m(t)/t$ When μ grows arbitrary large, results follow from liminf of $m(t)/t$, where RHS is zero. □

1.5 Central limit theorem for renewal processes

Theorem 1.10. *Let X_n be iid random variables with $\mu = \mathbb{E}[X_n] < \infty$ and $\sigma^2 = Var(X_n) < \infty$. Then*

$$\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} \to^d N(0,1)$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N(t) < u\} \iff \{S_u > t\}$. By equating probability measures on both sides, we get

$$\Pr\{N(t) < u\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t - u\mu}{\sigma\sqrt{u}}\right\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2\right\}.$$

By central limit theorem, $\frac{S_u - u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. Also, note that

$$\lim_{t\to\infty} -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2 = -y.$$

These results combine with the symmetry of normal random variable to give us the result. □

A Wald's Lemma

An integer random variable T is called a **stopping time** with respect to the *independent* random sequence $\{X_n : n \in \mathbb{N}\}$ if the event $\{N = n\}$ depends only on $\{X_1, \cdots, X_n\}$ and is independent of $\{X_{n+1}, X_{n+2}, \cdots\}$.

Intuitively, if we observe the X_n's in sequential order and N denotes the number observed before stopping then. Then, we have stopped after observing, $\{X_1, \ldots, X_N\}$, and before observing $\{X_{N+1}, X_{N+2}, \ldots\}$. The intuition behind a stopping time is that it's value is determined by past and present events but NOT by future events.

1. For instance, while traveling on the bus, the random variable measuring "Time until bus crosses Majestic and after that one stop" is a stopping time as it's value is determined by events before it happens. On the other hand "Time until bus stops before Majestic is reached" would not be a stopping time in the same context. This is because we have to cross this time, reach Majestic and then realize we have crossed that point.

2. Consider $X_n \in \{0,1\}$ *iid* Bernoulli$(1/2)$. Then $N = \min\{n \in \mathbb{N} : \quad \sum_{i=1}^n X_i = 1\}$ is a stopping time. For instance, $\Pr\{N = 2\} = \Pr\{X_1 = 0, X_2 = 1\}$ and hence N is a stopping time by definition.

3. **Random Walk Stopping Time** Consider X_n *iid* bivariate random variables with

$$\Pr\{X_n = 1\} = \Pr\{X_n = -1\} = \frac{1}{2}.$$

Then $N = min\{n \in \mathbb{N} : \quad \sum_{i=1}^n X_i = 1\}$ is a stopping time.

A.1 Properties of stopping time

Let N_1, N_2 be two stopping times with respect to independent random sequence $\{X_i : i \in \mathbb{N}\}$ then,

i. $N_1 + N_2$ is a stopping time.

ii. $\min\{N_1, N_2\}$ is a stopping time.

Proof. Let $\{X_i : i \in \mathbb{N}\}$ be an independent random sequence, and N_1, N_2 associated stopping times.

i. It suffices to show that the event $\{N_1 + N_2 = n\}$ depends only on random variables $\{X_1, \ldots, X_n\}$ and independent of $\{X_{n+1}, \ldots\}$. To this end, we observe that

$$\{N_1 + N_2 = n\} = \bigcup_{k=0}^{n} \{N_1 = k, N_2 = n-k\}.$$

Result follows since the events $\{N_1 = k\}$ and $\{N_2 = n-k\}$ depend solely on $\{X_1, \ldots, X_n\}$ for all $k \in \{0, \ldots, n\}$.

ii. It suffices to show that the event $\min\{N_1, N_2\} > n\}$ depends solely on $\{X_1, \ldots, X_n\}$.

$$\min\{N_1, N_2\} > n\} = \{N_1 > n\} \cap \{N_2 > n\}.$$

The result follows since the events $\{N_1 > n\}$ and $\{N_2 > n\}$ depend solely on $\{X_1, \ldots, X_n\}$.

□

Lemma A.1 (Wald's Lemma). *Let $\{X_i : \quad i \in \mathbb{N}\}$ be iid random variables with finite mean $\mathbb{E}[X_1]$ and let N be a stopping time with respect to this set of variables, such that $\mathbb{E}[N] < \infty$. Then,*

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[X_1]\mathbb{E}[N].$$

Proof. We first show that the event $\{N \geq n\}$ is independent of X_k, for any $k \geq n$. To this end, observe that

$$\{N \geq k\} = \{N < k\}^c = \{N \leq k-1\}^c = \left(\bigcup_{i=1}^{k-1} \{N = i\} \right)^c.$$

Recall that N is a stopping time and the event $\{N = i\}$ depends only on $\{X_1, \dots, X_i\}$, by definition. Therefore, $\{N \geq k\}$ depends only on $\{X_1, \dots, X_{k-1}\}$, and is independent of the future and present samples. Hence, we can write the Nth renewal instant for a stopping time N as

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}\left[\sum_{n\in\mathbb{N}} X_n 1_{\{N\geq n\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}X_n \mathbb{E}\left[1_{\{N\geq n\}}\right] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n\in\mathbb{N}} 1_{\{N\geq n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[N].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it since the summand is positive and we apply monotone convergence theorem. □

Proposition A.2 (Wald's Lemma for Renewal Process). *Let $\{X_n, n \in \mathbb{N}\}$ be iid inter-arrival times of a renewal process $N(t)$ with $\mathbb{E}[X_1] < \infty$, and let $m(t) = \mathbb{E}[N(t)]$ be its renewal function. Then, $N(t)+1$ is a stopping time and*

$$\mathbb{E}\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mathbb{E}[X_1][1 + m(t)].$$

Proof. It is easy to see that $\{N(t)+1 = n\}$ depends solely on $\{X_1, \dots, X_n\}$ from the discussion below.

$$\{N(t)+1 = n\} \iff \{S_{n-1} \leq t < S_n\} \iff \left\{\sum_{i=1}^{n-1} X_i \leq t < \sum_{i=1}^{n-1} X_i + X_n\right\}.$$

Thus $N(t)+1$ is a stopping time, and the result follows from Wald's Lemma. □