Lecture 08: Key Renewal Theorem and Applications

1 Key Renewal Theorem

For each $\delta > 0$ and $n \in \mathbb{N}$, we define intervals $I(n,\delta) = [(n-1)\delta, n\delta)$ that partition the positive axis $\mathbb{R}_+ = [0,\infty)$. Let $h : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function bounded over finite intervals, denoting

$$\underline{m}(h,n,\delta) = \inf\{h(u) : u \in I(n,\delta)\} \qquad \overline{m}(h,n,\delta) = \sup\{h(u) : u \in I(n,\delta)\}.$$

A function $h : \mathbb{R}_+ \mapsto \mathbb{R}$ is **directly Riemann integrable** and denoted by $h \in \mathbb{D}$ if the partial sums obtained by summing the infimum and supremum of h, taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is,

$$\overline{\sigma}_\delta \triangleq \lim_{\delta\to 0} \delta \sum_{n\in\mathbb{N}} \overline{m}(h,n,\delta) = \lim_{\delta\to 0} \delta \sum_{n\in\mathbb{N}} \underline{m}(h,n,\delta) \triangleq \underline{\sigma}_\delta.$$

If both limits exist and are equal, then the integral value is equal to the limit.

We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive M, a function $g : [0,M] \to \mathbb{R}$ is Riemann integrable if

$$\lim_{\delta\to 0} \delta \sum_{n\le M/\delta} \overline{m}(g,n,\delta) = \lim_{\delta\to 0} \delta \sum_{n\le M/\delta} \underline{m}(g,n,\delta).$$

In this case, the limit is the value of the integral. For h defined on $\mathbb{R}_+$,

$$\int_{u\in\mathbb{R}_+} h(u)du = \lim_{M\to\infty} \int_0^M h(u)du,$$

if the limit exists. For many functions, this limit may not exist.

A directly Riemann integrable function over $\mathbb{R}_+$ is also Riemann integrable, but the converse need not be true. For instance, consider the following Riemann integrable function

$$h(t) = \sum_{n\in\mathbb{N}} 1\left\{t \in \left[n - \frac{1}{(2n^2)}, n + \frac{1}{(2n^2)}\right]\right\}$$

is Riemann integrable, but $\delta \sum_{n\in\mathbb{N}} \overline{m}(h,n,\delta)$ is always infinite for every $\delta > 0$.

Proposition 1.1. *Following are sufficient conditions for a function h to be directly Riemann integrable.*

(a) If h is non-negative, continuous, and has finite support.

(b) If h is non-negative, continuous, bounded, and $\overline{\sigma}_\delta$ is bounded for some δ.

(c) If h is non-negative, monotone non-increasing, and Riemann integrable.

(d) *If h is non-negative and bounded above by a directly Riemann integrable function.*

Proposition 1.2 (Tail Property). *If h is non-negative, directly Riemann integrable, and has bounded integral value, then*

$$\lim_{t\to\infty} h(t) = 0.$$

Theorem 1.3 (Key Renewal Theorem). *Consider a renewal process with renewal function $m(t)$, and the mean and the distribution of inter-renewal times being denoted by μ and F respectively. If F is non-lattice and $F(\infty) = 1$, then for any $h \in \mathbb{D}$, we have*

$$\lim_{t\to\infty} \int_0^\infty h(t-x)dm(x) = \frac{1}{\mu}\int_0^\infty h(t)dt. \tag{1}$$

If F is lattice with period d and $\sum_{k\in\mathbb{N}_0} h(t+kd)$ converges, then

$$\lim_{n\to\infty} \int_0^\infty h(t+nd-x)dm(x) = \frac{d}{\mu}\sum_{n\in\mathbb{N}_0} h(t+kd).$$

Proposition 1.4 (Equivalence). *Blackwell's theorem and key renewal theorem are equivalent.*

Proof. Let's assume key renewal theorem is true. We select h as a simple function with value unity on interval $[0,a]$ and zero elsewhere. That is,

$$h(x) = 1_{\{x\in[0,a]\}}.$$

It is easy to see that this function is directly Riemann integrable.

To see how we can prove the key renewal theorem from Blackwell's theorem, observe from Blackwell's theorem that,

$$\lim_{t\to\infty}\frac{dm(t)}{dt} \stackrel{(a)}{=} \lim_{a\to 0}\lim_{t\to\infty}\frac{m(t+a)-m(t)}{a} = \frac{1}{\mu}.$$

where in (a) we can exchange the order of limits under certain regularity conditions. We defer the formal proof for a later stage. ☐

Key renewal theorem is very useful in computing the limiting value of some function $g(t)$, probability or expectation of an event at an arbitrary time t, for a renewal process. This value is computed by conditioning on the time of last renewal prior to time t.

2 Alternating renewal processes

Alternating renewal processes form an important class of renewal processes, and model many interesting applications. We find one natural application of key renewal theorem in this section.

Let $\{(Z_n,Y_n),\ n \in \mathbb{N}\}$ be an *iid* random process, where Y_n and Z_n are not necessarily independent. A renewal process where each inter-renewal time T_n consist of **on** time Z_n followed by **off** time Y_n, is called an **alternating renewal process**. We denote the distributions for on, off, and renewal periods by H, G, and F, respectively. Let

$$P(t) = \Pr\{\text{on at time } t\}.$$

To see that the alternating renewal process is indeed a renewal process, it needs to be established that $\{T_n : n \in \mathbb{N}\}$ is an *iid* sequence. However, this follows trivially from the fact that $\{f(Y_n,Z_n) : n \in \mathbb{N}\}$ is an *iid* sequence whenever $\{(Z_n, Y_n),\ n \in \mathbb{N}\}$ is an *iid* sequence. Let $f(a,b) = a+b$ to see that $\{T_n = Y_n + Z_n : n \in \mathbb{N}\}$ is an *iid* sequence.

For the renewal process with nth inter-renewal time T_n for each $n \in \mathbb{N}$, the nth renewal instant is denoted by $S_n = \sum_{i=1}^{n} T_n$. We can define a stochastic process $\{W(t) \in \{0,1\}, t \geqslant 0\}$ that takes values 1 and 0, when the renewal process is in on and off state respectively. In particular, we can write

$$W(t) = 1\{A(t) < Y_{N(t)+1}\}.$$

It is easy to see that W is a regenerative process with regenerative sequence S.

Theorem 2.1 (on probability). *Let m be the renewal function associated with the renewal process $\{S_n : n \in \mathbb{N}\}$ with a non-lattice inter-renewal duration distribution F. If $\mathbb{E}[Z_n + Y_n] < \infty$, then*

$$P(t) = \bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y).$$

Proof. We recall that $P(t) = P\{W(t) = 1\}$ and compute the kernel

$$P\{W(t) = 1, T_1 > t\} = P\{Z_1 > t, T_1 > t\} = P\{Z_1 > t\} = \bar{H}(t).$$

Hence, we can write the renewal equation for $P(t)$ as

$$P(t) = \bar{H}(t) + (F * P)(t).$$

Result follows from the solution of renewal equation. □

Corollary 2.2 (limiting on probability). *If $\mathbb{E}[Z_n + Y_n] < \infty$ and F is non-lattice, then*

$$\lim_{t\to\infty} P(t) = \frac{\mathbb{E}[Z_n]}{\mathbb{E}[Y_n] + \mathbb{E}[Z_n]}.$$

Proof. Since H is the distribution function of the non-negative random variable Z_n, it follows that

$$\lim_{t\to\infty} \bar{H}(t) = 0, \text{ and } \int_0^\infty \bar{H}(t)dt = \mathbb{E}Z_n.$$

Applying key renewal theorem to Theorem 2.1, we get the result. □

2.1 Age and excess times

Consider a renewal process with renewal instants $\{S_n : n \in \mathbb{N}\}$, and *iid* inter-renewal times $\{X_n : n \in \mathbb{N}\}$ with the common non-lattice distribution F. At time t, the last renewal occurred at time $S_{N(t)}$, and the next renewal will occur at time $S_{N(t)+1}$. Recall that the age $A(t)$ is the time since the last renewal and the excess time $Y(t)$ is the time till the next renewal. That is,

$$A(t) = t - S_{N(t)}, \qquad Y(t) = S_{N(t)+1} - t.$$

We are interested in finding the limiting distribution of age and excess times. That, is for a fixed x, we wish to compute

$$\lim_{t\to\infty} \Pr\{A(t) \le x\}, \qquad \lim_{t\to\infty} \Pr\{Y(t) \le x\}.$$

Proposition 2.3. *Limiting age distribution for a renewal process with non-lattice distribution F is*

$$\lim_{t\to\infty} \Pr\{A(t) \le x\} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt.$$

Proof. We will call this renewal process to be on, when the age is less than x. That is, we consider an alternative renewal process W such that

$$W(t) = 1\{A(t) \leq x\}.$$

This is an alternating renewal process with finite probability of off times being zero. Further, we can write the nth on time Z_n for this renewal process as

$$Z_n = \min\{x, X_n\}.$$

From limiting on probability of alternating renewal process, we get

$$\lim_{t\to\infty} \Pr\{A(t) \leq x\} = \lim_{t\to\infty} P\{W(t) = 1\} = \frac{\mathbb{E}\min\{x, X\}}{\mathbb{E}X} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt.$$

□

Alternative proof. Another way of evaluating $\lim_{t\to\infty} \Pr\{A(t) \leq x\}$ is to note that $\{A(t) \leq x\} = \{S_{N(t)} \geq t - x\}$. From the distribution of $S_{N(t)}$ and the fact that the support of renewal function $m(t)$ is positive real life,

$$\Pr\{A(t) \leq x\} = \Pr\{S_{N(t)} \geq t - x\} = \int_{t-x}^{\infty} \bar{F}(t-y)dm(y) = \int_{-\infty}^{x} \bar{F}(y)dm(t-y) = \int_0^x \bar{F}(u)dm(t-u).$$

Applying key renewal theorem, we get the result. □

We see that the limiting distribution of age and excess times are identical. This can be observed by noting that if we consider the reversed processes (an identically distributed renewal process), $Y(t)$, the "excess life time" at t is same as the age at t, $A(t)$ of the original process.

Proposition 2.4. *Limiting excess time distribution for a renewal process with non-lattice distribution F is*

$$\lim_{t\to\infty} \Pr\{Y(t) \leq x\} = \frac{1}{\mu}\int_0^x \bar{F}(t)dt.$$

Proof. We can repeat the same proof for limiting age distribution, where we obtain an alternative renewal process by defining on times when the excess time is less than x. That is, we consider an alternative renewal process W such that

$$W(t) = 1\{Y(t) \leq x\}.$$

□

Corollary 2.5. *Limiting mean excess time for a renewal process with iid inter-renewal times $\{X_n : n \in \mathbb{N}\}$ having non-lattice distribution F and mean μ is*

$$\lim_{t\to\infty} \mathbb{E}Y(t) = \frac{\mathbb{E}[X^2]}{2\mu}.$$

Proof. One can get the limiting mean from the limiting distribution by integrating its complement. This involves exchanging limit and integration, which can be justified using monotone convergence theorem. Hence, exchanging integrals using Fubini's theorem and integrating by parts, we get

$$\lim_{t\to\infty} \mathbb{E}Y(t) = \int_0^\infty \lim_{t\to\infty} P\{Y(t) > x\}dx = \frac{1}{\mu}\int_0^\infty\int_x^\infty \bar{F}(t)dt = \frac{1}{2\mu}\int_0^\infty \bar{F}(t)dt^2 = \frac{1}{2\mu}\int_0^\infty t^2 dF(t).$$

Alternatively, one can derive it directly from the regenerative process theory. □

Lemma 2.6. *Limiting empirical time average of excess time for a renewal process with iid inter-renewal times $\{X_n : n \in \mathbb{N}\}$ having non-lattice distribution F and mean μ is almost surely*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)du = \frac{\mathbb{E}X^2}{2\mu}.$$

Proof. Recall that excess times are linearly decreasing in each renewal duration, with value X_n to 0 in nth renewal duration of length X_n. Conditioned on $N(t)$, one can write

$$\int_0^t Y(u)du = \frac{1}{2}\sum_{n=1}^{N(t)} X_n^2 + \int_{S_{N(t)}}^t (S_{N(t)+1} - u)du.$$

In particular, one can write the following

$$\frac{\sum_{n=1}^{N(t)} X_n^2}{2N(t)} \left(\frac{N(t)}{t}\right) \leq \frac{1}{t}\int_0^t Y(u)du \leq \frac{\sum_{n=1}^{N(t)+1} X_n^2}{2(N(t)+1)} \left(\frac{N(t)+1}{t}\right).$$

Result follows from strong law of large number by taking limits on both sides. □

2.2 The Inspection Paradox

Define $X_{N(t)+1} = A(t) + Y(t)$ as the length of the renewal interval containing t, in other words, the length of current renewal interval.

Theorem 2.7 (inspection paradox). *For any x, the length of the current renewal interval to be greater than x is more likely than that for an ordinary renewal interval with distribution function F. That is,*

$$P\{X_{N(t)+1} > x\} \geq \bar{F}(x).$$

Proof. Conditioning on the joint distribution of last renewal instant and number of renewals, we can write

$$\Pr\{X_{N(t)+1} > x\} = \int_0^t \Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} dF_{(S_{N(t)}, N(t))}.$$

Now we have,

$$\Pr\{X_{N(t)+1} > x | S_{N(t)} = y, N(t) = n\} = \Pr\{X_{n+1} > x | X_{n+1} > t-y\} = \frac{\Pr\{X_{n+1} > \max(x, t-y)\}}{\Pr\{X_{n+1} > t-y\}}.$$

Applying Chebyshev's sum inequality to increasing positive functions $f(z) = 1\{t > x\}$ and $g(z) = 1\{z > t-y\}$, we get

$$\mathbb{E}f(X_{n+1})g(X_{n+1})/\mathbb{E}g(X_{n+1}) \geq \mathbb{E}f(X_{n+1}) = \bar{F}(x).$$

We get the result by integrating over the joint distribution. □

We also have a weaker version of inspection paradox involving the limiting distribution of $X_{N(t)+1}$.

Lemma 2.8. *For any x, the limiting probability of length of the current renewal interval being greater than x is larger than the corresponding probability of an ordinary renewal interval with distribution function F. That is,*

$$\lim_{t\to\infty} P\{X_{N(t)+1} > x\} \geq \bar{F}(x).$$

Proof. Consider an alternating renewal process W for which the on time is the renewal duration if greater than x, and zero otherwise. That is,

$$W(t) = 1\{X_{N(t)+1} > x\}.$$

Hence, each renewal duration consists of either on or off intervals, depending if the renewal duration length is greater than x or not. We can denote nth on and off times by Z_n and Y_n respectively, where

$$Z_n = X_n 1\{X_n > x\}, \qquad Y_n = X_n 1\{X_n \leq x\}.$$

From the definition, we have

$$\mathbb{E}W(t) = P\{X_{N(t)+1} > x\} = P\{\text{ on at time } t\}.$$

From the alternating renewal process theorem, we conclude that

$$\lim_{t\to\infty} \Pr\{X_{N(t)+1} > x\} = \frac{\mathbb{E}X1\{X > x\}}{\mathbb{E}X}.$$

The result follows from Chebyshev's sum inequality applied to positive increasing function $f(z) = z$ and $g(z) = 1\{z > x\}$. □

The inspection paradox states, in essence, that if we pick a point t, it is more likely that an inter-renewal interval with larger length will contain t than the smaller ones. For instance, if X_i were equally likely to be ε or $1-\varepsilon$, we see that the mean of any inter arrival length is 1 for any value of $\varepsilon \in (0,1)$. However, for small values ε, it is more likely that a given t will be in an interval of length $1-\varepsilon$ than in an interval of length ε.

Proposition 2.9. *If the inter arrival time is non-lattice and $\mathbb{E}[X^2] < \infty$, we have*

$$\lim_{t\to\infty}\left(m(t) - \frac{t}{\mu}\right) = \frac{\mathbb{E}[X^2]}{2\mu^2} - 1.$$

Proof. From definition of excess time $Y(t)$ and Wald's lemma for stopping time $N(t)+1$ for renewal processes, it follows

$$\mathbb{E}S_{N(t)+1} = \mathbb{E}[\sum_{i=1}^{N(t)+1} X_n] = \mu(m(t)+1) = t + \mathbb{E}[Y(t)].$$

□

A Chebyshev's sum inequality

Lemma A.1. *Let $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R} \to \mathbb{R}_+$ be arbitrary functions with the same monotonicity. For any random variable X, functions $f(X)$ and $g(X)$ are positive and*

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

Proof. Let Y be a random variable independent of X and with the same distribution. Then,

$$(f(X) - f(Y))(g(X) - g(Y)) \geq 0.$$

Taking expectation on both sides the result follows. □