# Lecture 09: Age-Dependent Branching and Delayed Renewal

## 1 Age-dependent branching process

Suppose a population where each organism lives for an *iid* random time period of $X$ units with common distribution function $F$. Just before dying, each organism produces a number of offsprings $N$, an *iid* discrete random variable with common distribution $P$. Let $X(t)$ denote the number of organisms alive at time $t$. The stochastic process $\{X(t),\ t \geqslant 0\}$ is called an age-dependent branching process. Let $X(t)$ be the number of individuals alive at time $t$. We are interested in computing $M(t) = \mathbb{E}X(t)$ when $m = \mathbb{E}[N] = \sum_{j \in \mathbb{N}} jP_j$. This is a popular model in biology for population growth of various organisms.

**Theorem 1.1.** *If* $X(0) = 1$*,* $m > 1$ *and* $F$ *is non lattice, then*

$$\lim_{t \to \infty} e^{-\alpha t} M(t) = \frac{m-1}{m^2 \alpha \int_0^\infty x e^{-\alpha x dF(x)}},$$

*where* $\alpha > 0$ *is the unique solution to the equation* $\int_0^\infty e^{-\alpha x} dF(x) = \frac{1}{m}$.

*Proof.* Let $T_1$ and $N_1$ denote the life time and offsprings of the first organism. If $T_1 > t$, then $X(t)$ is still equal to $X(0) = 1$. If $T_1 \leq t$, then $X(T_1) = N_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time $T_1$. Hence, we can write $X(t) = \sum_{i=0}^{N_1} X_i(t - T_1)$ for this case. Therefore,

$$M(t) = \mathbb{E}[X(t)1\{T_1 > t\}] + \mathbb{E}[X(t)1\{T_1 \leq t\}] = \mathbb{E}[\int_t^\infty dF(u) + \int_0^t dF(u) \sum_{i=0}^{N_1} X_i(t-u)] = \bar{F}(t) + m \int_0^t M(t-u) dF(u).$$

This looks almost like a renewal function. Multiplying both sides of the above equation by $e^{-\alpha t}$, we get

$$M(t)e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + m \int_0^t e^{-\alpha(t-u)} e^{-\alpha u} dF(u).$$

If $dG(t) \triangleq m e^{-\alpha t} dF(t)$ was a density function on $\mathbb{R}_+$, then the above equation would exactly be a renewal equation, with the solution

$$M(t)e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + \int_0^t e^{-\alpha(t-u)} \bar{F}(t-u) dm_G(u).$$

Here, $m_G(t) = \sum_{n \in \mathbb{N}} G_n(t)$ is the renewal function associated with inter-renewal distribution $G$. Clearly, the $\alpha > 0$ that ensures $G$ is a distribution function is the unique solution to the equation

$$1 = G(1) = m \int_0^\infty e^{-\alpha t} dF(t).$$

Since $e^{-\alpha t}\bar{F}(t)$ is non-negative, monotone non-increasing and integrable, it directly Riemann integrable. Hence, we can apply key renewal theorem to the limiting value of solution to renewal equation to obtain

$$\lim_{t\to\infty} M(t)e^{-\alpha t} = \frac{1}{\mu_G}\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{\int_0^\infty e^{-\alpha t}\bar{F}(t)dt}{m\int_0^\infty xe^{-\alpha t}dF(t)}.$$

Result follows from integration by parts, and showing that

$$\int_0^\infty e^{-\alpha t}\bar{F}(t)dt = \frac{1}{\alpha} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha t}dF(t) = \frac{1}{\alpha}\left(1-\frac{1}{m}\right).$$

□

# 2 Delayed renewal process

Many times in practice, we have a "delayed start to a renewal process". That is, the arrival process has *iid* inter-arrival times $T_i$ for $i \geq 2$ with common distribution function $F$. Whereas, the first inter-arrival times $T_1$ is independent and has a different distribution $G$. The associated counting process is called a **delayed renewal process** and denoted by $\{N_D(t) : t \geqslant 0\}$. Let $S_0 = 0$ and $n$th arrival instant $S_n = \sum_{i=1}^n T_i$. Then, then following inverse relationship holds between counting and arrival process,

$$N_D(t) = \sup\{n \in \mathbb{N} : S_n \leq t\}.$$

## 2.1 Distribution functions

**Lemma 2.1.** *The distribution function of nth arrival instant $S_n$ is*

$$P\{S_n \leq t\} = (G * F_{n-1})(t).$$

**Lemma 2.2.** *The distribution function of counting process $N_D(t)$ is*

$$P\{N_D(t) = n\} = P(S_n \leq t) - P(S_{n+1} \leq t) = (G * F_{n-1})(t) - (G * F_n)(t).$$

**Lemma 2.3.** *Let $m(t)$ be the renewal function associated with a renewal process with inter-arrival distribution $F$. Then, the renewal function $m_D(t) = \mathbb{E}N_D(t)$ associated with the delayed renewal process $N_D(t)$ is*

$$m_D(t) = \mathbb{E}N_D(t) = \sum_{n\in\mathbb{N}} (G * F_{n-1})(t) = G(t) + (G * m)(t).$$

**Lemma 2.4.** *We denote the moment-generating function of a random variable with distribution $m_D(t), G(t), F(t)$ by $\tilde{m}_D(s), \tilde{G}, \tilde{F}$ respectively. Then,*

$$\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)}$$

## 2.2 Limit theorems

The limit theorems for delayed renewal process are identical to those for renewal processes.

**Lemma 2.5 (Basic renewal theorem).** $\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu_F}$.

**Lemma 2.6 (Elementary renewal theorem).** $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu_F}$.

**Lemma 2.7 (Blackwell's theorem).** *If the inter-renewal distribution $F$ is non-lattice, then*

$$\lim_{t\to\infty} m_D(t+a) - m_D(t) = \frac{a}{\mu_F}.$$

*If the distributions $F$ and $G$ are lattice with period $d$, then*

$$\lim_{n\in\mathbb{N}} \mathbb{E}[\text{Number of renewals at } nd] = \frac{d}{\mu_F}.$$

**Lemma 2.8 (Key renewal theorem).** *If $F$ is non-lattice, $\mu_F < \infty$ and $h \in \mathbb{D}$, then*

$$\lim_{t\to\infty} \int_0^t h(t-x)dm_D(x) = \frac{1}{\mu_F} \int_0^\infty h(t)dt.$$

### 2.3 Distribution of the last renewal time

Using the regenerative process theory for $s \leq t$, we can write

$$P\{S_{N(t)} \leq s\} = \bar{G}(t) + \int_0^s \bar{F}(t-u)dm(u).$$

# 3 Equilibrium renewal process

For $x \geq 0$, we can define the **equilibrium distribution** of $F$ as

$$F_e(x) = \frac{1}{\mu} \int_0^x \bar{F}(y)dy.$$

**Lemma 3.1.** *The moment generating function of $F_e(x)$ is*

$$\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu}.$$

*Proof.* By definition, $\tilde{F}_e(s) = \mathbb{E}\left[e^{-sX}\right]$, where $X$ is a random variable with distribution function $F_e(x)$. We use integration by parts, to write

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx} dF_e(x) = \frac{1}{s\mu} - \frac{1}{s\mu} \int_0^\infty e^{-sx} dF(x) = \frac{1}{s\mu}(1-\tilde{F}(s)).$$

□

A delayed renewal process with the initial arrival distribution $G = F_e$ is called the **equilibrium renewal process**. Observe that $F_e$ is the limiting distribution of the age and the excess time for the renewal process with common inter-renewal distribution $F$. Hence, if we start observing a renewal process at some arbitrarily large time $t$, then the observed renewal process is the equilibrium renewal process. This delayed renewal process exhibits stationary properties. That is, the limiting behaviors are exhibited for all times.

**Theorem 3.2 (renewal function).** *The renewal function $m_e(t)$ for the equilibrium renewal process is linear for all times. That is, $m_e(t) = \frac{t}{\mu}$.*

*Proof.* We know that the Laplace transform of renewal function $m_e(t)$ is given by

$$\tilde{m}_e(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{\tilde{F}_e(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu}.$$

Further, we know that the Laplace transform of function $t/\mu$ is given by

$$\mathcal{L}_{t/\mu}(s) = \frac{1}{\mu}\int_0^\infty e^{-sx}dx = \frac{1}{s\mu}.$$

Since moment generating function is a one-to-one map, $m_e(t) = \frac{t}{\mu}$ is the unique renewal function. □

**Theorem 3.3 (excess time).** *The distribution of excess time $Y_e(t)$ for the equilibrium renewal process is stationary. That is,*

$$P(Y_e(t) \le x) = F_e(x),\ t \ge 0.$$

*Proof.* Since the excess time $Y_e(t)$ is regenerative process and $dm_e(t) = 1/\mu$, we can write

$$P\{Y_e(t) > x\} = \bar{F}_e(t+x) + \frac{1}{\mu}\int_0^t \bar{F}(t+x-u)du = \bar{F}_e(t+x) + \frac{1}{\mu}\int_x^{t+x} \bar{F}(y)dy = \bar{F}_e(x).$$

□

**Theorem 3.4 (stationary increments).** *The counting process $\{N_e(t), t \geqslant 0\}$ for the equilibrium renewal process has stationary increments.*

*Proof.* When we start observing the process at time $s$, the observed renewal process is delayed renewal process with initial distribution being the original distribution. Hence, the number of renewals $N_e(t+s) - N_e(s) = N_e(t)$ in time interval of duration $t$ is shift invariant. □

## 3.1 Exponential renewal intervals

Consider the case, when inter-renewal time distribution $F$ for a delay renewal process is exponential with rate $\lambda$. Here, one would expect the equilibrium distribution $F_e = F$, since Poisson process has stationary and independent increments. We observe that

$$F_e(x) = \frac{1}{\mu}\int_0^x \bar{F}(y)dy = \lambda\int_0^x \mathrm{e}^{-\lambda y}dy = 1 - e^{-\lambda x} = F(x).$$

We see that $F_e$ is also distributed exponentially with rate $\lambda$. Indeed, this is a Poisson process with rate $\lambda$.