Lecture 10: Renewal Reward Processes

1 Renewal reward process

Consider a renewal process $\{N(t), t \geqslant 0\}$ with *iid* inter renewal times $\{X_n : n \in \mathbb{N}\}$ having common distribution F. The reward sequence $\{R_n : n \in \mathbb{N}\}$ consists of reward R_n at the end of nth renewal interval X_n for each $n \in \mathbb{N}$. Let (X_n, R_n) be *iid* with the reward R_n earned in nth renewal possibly dependent on the duration X_n. Then the **reward process** $\{R(t), t \geqslant 0\}$ consists of accumulated reward earned by time t as

$$R(t) = \sum_{i=1}^{N(t)} R_i.$$

Theorem 1.1. *Let* $\mathbb{E}[|R|]$ *and* $\mathbb{E}[|X|]$ *be finite.*

(a) $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$ *a.s.*

(b) $\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$.

Proof. We can write

$$\frac{R(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{t} = \left(\frac{\sum_{i=1}^{N(t)} R_i}{N(t)}\right)\left(\frac{N(t)}{t}\right).$$

(a) By the strong law of large numbers (almost sure convergence law) we obtain that,

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} = \mathbb{E}[R],$$

and by the basic renewal theorem (almost sure convergence law) we obtain that,

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}.$$

(b) Since $N(t)+1$ is a stopping time for the sequence $\{(X_1,R_1),(X_2,R_2),\dots\}$, by Wald's lemma,

$$\mathbb{E}[R(t)] = \mathbb{E}\left[\sum_{i=1}^{N(t)} R_i\right] = \mathbb{E}\left[\sum_{i=1}^{N(t)+1} R_i\right] - \mathbb{E}R_{N(t)+1} = (m(t)+1)\mathbb{E}R_1 - \mathbb{E}R_{N(t)+1}.$$

Defining $g(t) \triangleq \mathbb{E}[R_{N(t)+1}]$, using elementary renewal theorem, it suffices to show that

$$\lim_{t\to\infty} \frac{g(t)}{t} = 0.$$

Observe that $R_{N(t)+1}$ is a regenerative process with the regenerative sequence being the renewal instants. We can write the kernel function as

$$K(t) \triangleq \mathbb{E}[R_{N(t)+1}1\{X_1 > t\}] = \int_t^\infty \mathbb{E}[R_1|X_1 = u]dF(u) \leq \int_t^\infty \mathbb{E}[|R_1||X_1 = u]dF(u).$$

Using the solution to renewal function, we can write $g(t)$ in terms of renewal function $m(t)$ as

$$g(t) = K(t) + (m * K)(t).$$

From finiteness of $\mathbb{E}|R|$, it follows that $\lim_{t\to\infty} K(t) = 0$, and we can choose T such that $|K(u)| \leq \varepsilon$ for all $u \geq T$. Hence, for all $t \geq T$, we have

$$\begin{aligned}\frac{|g(t)|}{t} &\leq \frac{|K(t)|}{t} + \int_0^{t-T} \frac{|K(t-u)|}{t} dm(u) + \int_{t-T}^t \frac{|K(t-u)|}{t} dm(u)\\ &\leq \frac{\varepsilon}{t} + \frac{\varepsilon m(t-T)}{t} + \mathbb{E}[|R_1|]\frac{(m(t) - m(t-T))}{t}.\end{aligned}$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get

$$\limsup_{t\to\infty} \frac{|g(t)|}{t} \leq \frac{\varepsilon}{\mathbb{E}X}.$$

The result follows since $\varepsilon > 0$ was arbitrary.

□

Lemma 1.2. *Reward $R_{N(t)+1}$ at the next renewal has different distribution than R_1.*

Proof. Notice that $R_{N(t)+1}$ is related to $X_{N(t)+1}$ which is the length of the renewal interval containing the point t. We have seen that larger renewal intervals have a greater chance of containing t. That is, $X_{N(t)+1}$ tends to be larger than a ordinary renewal interval. Formally, □

Lemma 1.3. *Renewal reward theorem applies to a reward process $R(t)$ that accrues reward continuously over a renewal duration. The total reward in a renewal duration X_n remains R_n as before, with the sequence$\{(X_n, R_n) : n \in \mathbb{N}\}$ being iid.*

Proof. Let the process $R(t)$ denote the accumulated reward till time t, when the reward accrual is continuous in time. Then, it follows that

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.$$

Result follows from application of strong law of large numbers. □

1.1 Limiting empirical average of age and excess times

To determine the average value of the age of a renewal process, consider the following gradual reward process. We assume the reward rate to be equal to the age of the process at any time t, and

$$R(t) = \int_0^t A(u)du.$$

Observe that age is a linear increasing function of time in any renewal duration. In nth renewal duration, it increases from 0 to X_n, and the total reward $R_n = X_n^2/2$. Hence, we obtain from the renewal reward theorem

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(u)du = \frac{\mathbb{E}R_n}{\mathbb{E}X_n} = \frac{\mathbb{E}X^2}{2\mathbb{E}X}.$$

A similar analysis to calculate the average excess time, where the reward per cycle is $\int_0^X (X-t)dt$ gives

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)du = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}.$$

Since $X_{N(t)+1} = A(t) + Y(t)$, we see that its average value is given by

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_{N(u)+1}du = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}.$$

It can be shown, under certain regularity conditions, that

$$\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}] = \frac{\mathbb{E}[R_1 X_1]}{\mathbb{E}[X_1]}.$$

Defining a cycle reward to equal the cycle length, we have

$$\lim_{t\to\infty} \mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}.$$

We see that this limit is always greater than $\mathbb{E}[X]$, except when X is constant. Such a result was to be expected in view of the inspection paradox.