Lecture 11: Discrete Time Markov Chains

1 Introduction

We have seen that *iid* sequences are easiest discrete time processes. However, they don't capture correlation well. Hence, we look at the discrete time stochastic processes of the form

$$X_{n+1} = f(X_n, Z_{n+1}),$$

where $\{Z_n : n \in \mathbb{N}\}$ is an iid sequence, independent of initial state X_0. If $X_n \in E$ for all $n \in \mathbb{N}_0$, then E is called the **state space** of process X. We consider a countable state space, and if $X_n = i \in \mathbb{E}$, then we say that the process X is in state i at time n. For a countable set E, a stochastic process $\{X_n \in E, n \in \mathbb{N}_0\}$ is called a **discrete time Markov chain (DTMC)** if for all positive integers $n \in \mathbb{N}_0$ and all states $i_0, i_1, \dots, i_{n-1}, i, j \in E$, the process X satisfies the Markov property

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\}.$$

1.1 Homogeneous Markov chain

If for each $n \in \mathbb{N}_0$, we have $p_{ij}(n) \triangleq P\{X_{n+1} = j | X_n = i\} = p_{ij}$. That is, when the transition probability does not depend on n, the DTMC is **homogeneous** and the matrix $P = \{p_{ij} : i, j \in \mathbb{E}\}$ is called the **transition matrix**.

For all states $i, j \in E$, if a non-negative matrix $A \in \mathbb{R}_+^{E \times E}$ has the following property

$$a_{ij} \geq 0, \qquad \sum_{j \in E} a_{ij} \leq 1,$$

then it is called a **sub-stochastic** matrix. If the second property holds with equality, then it is called a **stochastic** matrix. If in addition, A^T is stochastic, then A is called **doubly stochastic**. Clearly, the transition matrix P is stochastic matrix.

1.2 Transition graph

A transition matrix P is sometimes represented by a directed graph $G = (E, \{[i, j) \in E \times E : p_{ij} > 0\})$. In addition, this graph has a weight p_{ij} on each edge $e = [i, j)$.

2 Chapman Kolmogorov equations

We can define n-step transition probabilities for $i, j \in E$ and $m, n \in \mathbb{N}$

$$p_{ij}^{(n)} \triangleq P\{X_{n+m} = j | X_m = i\}.$$

It follows from the Markov property and law of total probability that

$$p_{ij}^{(m+n)} = \sum_{k\in E} p_{ik}^{(m)} p_{kj}^{(n)}.$$

We can write this result compactly in terms of transition probability matrix P as $P^{(n)} = P^n$. Let $\nu \in \mathbb{R}_+^E$ is a probability vector such that

$$\nu_n(i) = P\{X_n = i\}.$$

Then, we can write this vector ν_n in terms of initial probability vector ν_0 and the transition matrix P as

$$\nu_n = \nu_0 P^n.$$

2.1 Strong Markov property (SMP)

Let T be an integer valued stopping time with respect to the stochastic process X such that $P\{T < \infty\} = 1$. Then for all $i_0, \dots, i_{n-1}, \dots, i, j \in E$, the process X satisfies the **strong Markov property** if

$$P\{X_{T+1} = j | X_T = i, \dots, X_0 = i_0\} = P\{X_{T+1} = j | X_T = i\}.$$

Lemma 2.1. *Markov chains satisfy the strong Markov property.*

Proof. Let X be a Markov chain and $A = \{X_T = i, \dots, X_0 = i_0\}$. Then, we have

$$P\{X_{T+1} = j|A\} = \frac{\sum_{n\in\mathbb{N}_0} P\{X_{T+1} = j, A, T = n\}}{P\{A\}} = \sum_{n\in\mathbb{N}_0} p_{ij} \frac{P(A, T = n)}{P(A)} = p_{ij}.$$

This equality follows from the fact that $\{T = n\}$ is completely determined by $\{X_0, \dots, X_n\}$ □

As an exercise, if we try to use the Markov property on arbitrary random variable T, the SMP may not hold. For example, define a non-stopping time T for $j \in E$

$$T = \inf\{n \in \mathbb{N}_0 : X_{n+1} = j\}.$$

In this case, we have

$$P\{X_{T+1} = j | X_T = i, \dots, X_0 = i_0\} = 1\{p_{ij} > 0\} \neq P\{X_1 = j | X_0 = i\} = p_{ij}.$$

A useful application of the strong Markov property is as follows. Let $i_0 \in E$ be a fixed state and $\tau_0 = 0$ Let τ_n denote the stopping times at which the Markov chain visits i_0 for the nth time. That is,

$$\tau_n = \inf\{n > \tau_{n-1} : X_n = i_0\}.$$

Then $\{X_{\tau_n+m} : m \in \mathbb{N}_0\}$ is a stochastic replica of $\{X_m : m \in \mathbb{N}_0\}$ with $X_0 = i_0$ and can be studied as a regenerative process.

3 Communicating classes

State $j \in E$ is said to be **accessible** from state $i \in E$ if $p_{ij}^{(n)} > 0$ for some $n \in \mathbb{N}$, and denoted by $i \to j$. If two states $i, j \in E$ are accessible to each other, they are said to **communicate** with each other, denoted by $i \leftrightarrow j$. A set of states that communicate are called a **communicating class**.

Proposition 3.1. *Communication is an equivalence relation.*

Proof. Reflexivity and Symmetry are obvious. For transitivity, suppose $i \leftrightarrow j$ and $j \leftrightarrow k$. Suppose $p_{ij}^{(m)} > 0$ and $p_{jk}^{(n)} > 0$. Then by Chapman Kolmogorov, we have

$$p_{ik}^{(m+n)} = \sum_{l \in \mathbb{N}_0} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0$$

Hence transitivity is assured. □

3.1 Irreducibility and periodicity

A consequence of the previous result is that communicating classes are disjoint or identical. A Markov chain with a single class is called an **irreducible** Markov chain. The **period** of state i is defined as

$$d(i) = \gcd\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\}$$

If the period is 1, we say the state is **aperiodic**.

Proposition 3.2. *If $i \leftrightarrow j$, then $d(i) = d(j)$. Basically, periodicity is a class property.*

Proof. Let m and n be such that $p_{ij}^{(m)} p_{ji}^{(n)} > 0$. Suppose $p_{ii}^{(s)} > 0$. Then

$$p_{jj}^{(n+m)} \geq p_{ji}^{(n)} p_{ij}^{(m)} > 0$$

$$p_{jj}^{(n+s+m)} \geq p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} > 0$$

Hence $d(j)|n+m$ and $d(j)|n+s+m$ which implies $d(j)|s$. Hence $d(j)|d(i)$. By symmetrical arguments, we get $d(i)|d(j)$. Hence $d(i) = d(j)$. □

3.2 Transient and recurrent states

Let $f_{ij}^{(n)}$ denote the probability that starting from state i, the first transition into state j happens at time n. Then let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

Here f_{ij} would therefore denote the probability of ever entering state j given that we start at state i. State j is said to be **transient** if $f_{jj} < 1$ and **recurrent** if $f_{jj} = 1$.

Proposition 3.3. *The total number of visits to a state $j \in E$ is denoted by $N_j = \sum_{n \in \mathbb{N}_0} 1\{X_n = j\}$. Then, for each $m \in \mathbb{N}$ and $i \neq j$, we have*

$$P_j\{N_j = m\} = f_{jj}^{m-1}(1 - f_{jj}),\ m \in \mathbb{N}$$
$$P_i\{N_j = m\} = \begin{cases} 1 - f_{ij} & m = 0, \\ f_{ij} f_{jj}^{m-1}(1 - f_{jj}) & m \in \mathbb{N}. \end{cases}$$

Proof. For each $k \in \mathbb{N}$, the time of the kth visit to the state j is a stopping time. From strong Markov property, the next return to state j is independent of the past. Hence, each return to state j is an *iid* Bernoulli random variable with probability f_{jj}. It follows that the number of visits j is the time for first failure to return. Conditioned on $X_0 = j$, the distribution of N_j is geometric random variable with success probability $1 - f_{jj}$.

Conditioned on $X_0 = i$, the stopping time of first visit to j is a Bernoulli random variable with probability f_{ij}. Hence, the second result follows. □

Corollary 3.4. *For a Markov chain X,* $P_j\{N_j < \infty\} = 1\{f_{jj} < 1\}$.

Proof. We can write the event $\{N_j < \infty\}$ as disjoint union of events $\{N_j = n\}$, to get

$$P_j\{N_j \in \mathbb{N}\} = \sum_{n\in\mathbb{N}} P_j\{N_j = n\} = 1\{f_{jj} < 1\}.$$

□

In particular, this corollary implies the following

1. A transient state is visited a finite amount of times almost surely.
2. A recurrent state is visited infinitely often almost surely.
3. In a finite state Markov chain, not all states may be transient.

Proposition 3.5. *A state j is recurrent iff* $\sum_{k\in\mathbb{N}} p_{jj}^{(k)} = \infty$.

Proof. For any state $j \in E$, we can write

$$p_{ii}^{(k)} = \mathbb{P}_i\{X_k = i\} = \mathbb{E}_i 1\{X_k = i\}.$$

Using monotone convergence theorem to exchange expectation and summation, we obtain

$$\sum_{k\in\mathbb{N}} p_{ii}^{(k)} = \mathbb{E}_i \sum_{k\in\mathbb{N}} 1\{X_k = i\} = \mathbb{E}_i N_i.$$

Thus, $\sum_{k\in\mathbb{N}} p_{ii}^{(k)}$ represents the expected number of returns $\mathbb{E}_i N_i$ to a state i starting from state i, which we know to be finite if the state is transient and infinite if the state is recurrent. □

Proposition 3.6. *Transience and recurrence are class properties.*

Proof. Let us start with proving recurrence is a class property. Let i be a recurrent state and let $i \leftrightarrow j$. Hence there exist some $m, n > 0$, such that $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$. As a consequence of the recurrence, $\sum_{s\in\mathbb{N}_0} p_{ii}^{(s)} = \infty$. It follows that j is recurrent by observing

$$\sum_{s\in\mathbb{N}_0} p_{jj}^{(m+n+s)} \geq \sum_{s\in\mathbb{N}_0} p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} = \infty.$$

Now, if i were transient instead, we conclude that j is also transient by the following observation

$$\sum_{s\in\mathbb{N}_0} p_{jj}^{(s)} \leq \frac{\sum_{s\in\mathbb{N}_0} p_{ii}^{(m+n+s)}}{p_{ji}^{(n)} p_{ij}^{(m)}} < \infty.$$

□

Corollary 3.7. *If j is recurrent, then for any state i such that* $i \leftrightarrow j$, $f_{ij} = 1$.