Lecture 12 : Limit Theorems for Markov Chains

1 Limit Theorems

Let $N_j(t)$ denote the number of transitions into state $j \in E$ up to time t. That is,

$$N_j(t) = \sum_{k=1}^{t} 1\{X_k = j\}.$$

Let $S_0 = 0$, then we can define the nth arrival instants of state j as a stopping time

$$S_n(j) = \inf\{k > S_{n-1}(j) : X_k = j\}.$$

From strong Markov property it follows that X is a regenerative process with regenerative sequence $S(j) = \{S_n(j) : n \in \mathbb{N}\}$. We can define the inter-renewal duration, the number of time steps to return to the state j as

$$T_n(j) = S_n(j) - S_{n-1}(j).$$

If $X_0 = j$ and j is recurrent, then $S(j)$ is a renewal process with the *iid* inter-arrival distribution,

$$P_j\{T_1(j) = k\} = f_{jj}^{(k)},\ k \in \mathbb{N}.$$

Let $\mu_{jj} = \mathbb{E}_j T_1(j)$ be the mean inter-arrival time for the renewal process. Then,

$$\mu_{jj} = \begin{cases} \infty & j \text{ transient}, \\ \sum_{k\in\mathbb{N}} k f_{jj}^{(k)} & j \text{ recurrent}. \end{cases}$$

If $X_0 = i \neq j$, for some $i \leftrightarrow j$ and j recurrent, then $S(j)$ is a delayed renewal process with first inter-arrival distribution

$$P_i\{T_1(j) = k\} = f_{ij}^{(k)},\ k \in \mathbb{N}.$$

The associated counting process $N_j(t)$ has the inverse relationship with the renewal process $S(j)$. From the renewal theory, we have the following results.

Proposition 1.1 (basic renewal theorem). *If $i \leftrightarrow j$, then $P_i\left\{\lim_{n\in\mathbb{N}} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}\right\} = 1$.*

Proposition 1.2 (elementary renewal theorem). *If $i \leftrightarrow j$, then*

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{ij}^{(k)}}{n} = \lim_{n\in\mathbb{N}} \frac{\mathbb{E}_i[N_j(n)]}{n} = \frac{1}{\mu_{jj}}.$$

Proposition 1.3 (Blackwell's theorem). *If j is aperiodic (i.e., $d(j) = 1$), then*

$$\lim_{n\in\mathbb{N}} p_{ij}^{(n)} = \lim_{n\in\mathbb{N}} \mathbb{E}_i[\#\text{ renewals at } n] = \frac{1}{\mu_{jj}}$$

If j is periodic with period d, then

$$\lim_{n\in\mathbb{N}} p_{ij}^{(nd)} = \lim_{n\in\mathbb{N}} \mathbb{E}_i[\#\text{ renewals at } nd] = \frac{d}{\mu_{jj}}.$$

2 Positive and Null recurrence

A recurrent state j is said to be **positive recurrent** if $\mu_{jj} < \infty$ and **null recurrent** if $\mu_{jj} = \infty$. Let

$$\pi_j \triangleq \lim_{n\in\mathbb{N}} p_{jj}^{(nd)},$$

where d is the period of state j. Then $\pi_j > 0$ if and only if j is positive recurrent and $\pi_j = 0$ if j is null-recurrent.

Proposition 2.1. *Positive recurrence and null recurrence are class properties.*

An state that is aperiodic and positive recurrent is called **ergodic**. For a homogeneous Markov chain on state space E with transition probability matrix E, a probability distribution $\{\pi_j : j \in E\}$ is said to be **stationary** if for all states $j \in E$

$$\pi_j = \sum_{k\in E} \pi_k P_{kj}.$$

More compactly, π is stationary if $\pi = \pi P$.

Observe that for a Markov chain starting with its stationary distribution, then the distribution remains invariant for all times. That is, if π is the stationary distribution, and the Markov chain has initial distribution $\nu(0) = \pi$ at time 0, then at any time $n \in \mathbb{N}$, the Markov chain has distribution $\nu(n) = \pi$. Moreover since X_n has discrete states in E, the finite collection $(X_n, X_{n+1}, ...X_{n+m})$ have the same joint distribution. Hence it is a stationary process, and for all $k, m \in \mathbb{N}, i \in E^k$

$$P\{X_1 = i_1, \dots, X_k = i_k\} = P\{X_{m+1} = i_1, \dots, X_{m+k} = i_k\}.$$

Theorem 2.2. *An irreducible, aperiodic Markov Chain with countable state space E is of one of the following types.*

i) All the states are either transient or null recurrent. For all states $i, j \in E$,

$$\lim_{n\in\mathbb{N}} P_{ij}^n = 0,$$

and there exists no stationary distribution.

ii) All the states are positive recurrent, and hence the chain is ergodic. There exists a unique stationary distribution $\pi \in \Delta(E)$, defined for all $i, j \in E$

$$\pi_j \triangleq \lim_{n\in\mathbb{N}} P_{ij}^n > 0.$$

Proof. Let $\{X_n : n \in \mathbb{N}\}$ be an irreducible, aperiodic Markov chain with countable state space E.

i) Suppose that all states are either transient or null recurrent. Note that exactly one of these will hold since there is only one communicating class. This implies that $\mu_{jj} = \infty$ for each state $j \in E$, and it follows from Blackwell's theorem applied to renewals for Markov chains that for any states $i, j \in E$

$$\lim_{n \in \mathbb{N}} P_{ij}^{(n)} = \frac{1}{\mu_{jj}} = 0.$$

If there existed a stationary distribution $\pi \in \Delta(E)$ in this case. For any step size $n \in \mathbb{N}$ and states $i, j \in E$, we would then have

$$\pi_j = \sum_{i \in E} \pi_i P_{ij}^{(n)}, \qquad\qquad P_{ij}^{(n)} \leq 1.$$

We can change limits and summation using dominated convergence theorem, to get for any $j \in E$

$$\pi_j = \sum_{i \in E} \pi_i \lim_{n \in \mathbb{N}} P_{ij}^{(n)} = 0.$$

This contradicts π being a stationary distribution, proving the first part of the theorem.

ii) We assume that all states are positive recurrent. From the theorem hypothesis, elementary renewal theorem, and positive recurrence, we get

$$\pi_j = \lim_{n \in \mathbb{N}} P_{ij}^{(n)} = 1/\mu_{jj} > 0.$$

Further, for any finite set $A \subseteq E$, we have

$$\sum_{j \in A} P_{ij}^{(n)} \leq \sum_{j \in E} P_{ij}^{(n)} = 1.$$

Taking limit $n \in \mathbb{N}$ on both sides, we conclude that $\sum_{j \in A} \pi_j \leq 1$ for all A finite. Taking limit with respect to increasing sets $A \uparrow E$, we conclude,

$$\sum_{j \in E} \pi_j \leq 1.$$

Further, we can write for all $A \subseteq E$,

$$P_{ij}^{n+1} = \sum_{k \in E} P_{ik}^n P_{kj} \geq \sum_{k \in A} P_{ik}^n P_{kj}.$$

Applying limit $n \in \mathbb{N}$ on both sides, we get $\pi_j \geq \sum_{k \in A} \pi_k P_{kj}$ for all A finite. Hence, taking limits with respect to increasing sets $A \uparrow E$, we get for all state $j \in E$,

$$\pi_j \geq \sum_{k \in E} \pi_k P_{kj}.$$

Assuming that the inequality is strict for some state $j \in E$, we can sum the inequalities over all states $j \in E$. Since, summands are non-negative we can exchange summation orders to get

$$\sum_{j \in E} \pi_j > \sum_{j \in E} \sum_{k \in E} \pi_k P_{kj} = \sum_{k \in E} \pi_k \sum_{j \in E} P_{kj} = \sum_{k \in E} \pi_k.$$

This is a contradiction. Therefore, for any state $j \in E$

$$\pi_j = \sum_{k \in E} \pi_k P_{kj}.$$

Defining normalized $w_j = \frac{\pi_j}{\sum_{k\in I}\pi_k}$, we see that $\{w_j : j \in E\}$ is a stationary distribution and so at least one stationary distribution exists. If the initial distribution of this positive recurrent Markov chain is a stationary distribution $\{\lambda_j : j \in E\}$, then for any finite subset $A \subseteq E$, we get

$$\lambda_j = \Pr\{X_n = j\} = \sum_{i\in E} P^n_{ij}\lambda_i \geq \sum_{i\in A} P^n_{ij}\lambda_i.$$

As before, we take limit $n \in \mathbb{N}$, followed by limit of increasing subsets $A \uparrow E$, to obtain

$$\lambda_j \geq \sum_{i\in E} \pi_j \lambda_i = \pi_j.$$

To show $\lambda_j \leq \pi_j$, we use the fact that $P^n_{ij} \leq 1$. Let $A \subseteq E$ be a finite subset, then

$$\lambda_j = \sum_{i\in I} P^n_{ij}\lambda_i = \sum_{i\in A} P^n_{ij}\lambda_i + \sum_{i\notin A} P^n_{ij}\lambda_i \leq \sum_{i\in A} P^n_{ij}\lambda_i + \sum_{i\notin A} \lambda_i.$$

Using our standard approach of taking limit $n \in \mathbb{N}$, followed by $A \uparrow E$, we obtain

$$\lambda_j \leq \sum_{i\in E} \pi_j \lambda_i = \pi_j.$$

□

Corollary 2.3. *An irreducible, aperiodic Markov chain defined on a finite state space E will be positive recurrent.*

Proof. Suppose that the Markov chain is not positive recurrent, then

$$\lim_{n\in\mathbb{N}} P^{(n)}_{ij} = 0.$$

Interchanging limit and finite summation gives

$$0 = \sum_{j\in E} \lim_{n\in\mathbb{N}} P^{(n)}_{ij} = \lim_{n\in\mathbb{N}} \sum_{j\in E} P^{(n)}_{ij} = 1.$$

This is a contradiction. Hence the above mentioned chain is positive recurrent. □

Corollary 2.4. *For an irreducible and aperiodic Markov chain with stationary distribution π on countable state space E, we have*

$$\mathbb{E}_j[T_1(j)] \triangleq \mathbb{E}[T_1(j)|X_0 = j] = \frac{1}{\pi_j},\ j \in E.$$

Further, we can define the number of visits to state i during one renewal duration $S_1(j)$ as

$$N_i(S_1(j)) = \sum_{k=1}^{S_1(j)} 1\{X_n = i\}.$$

Proposition 2.5. *For an aperiodic and irreducible Markov chain X with stationary distribution π on countable state space E, the mean number of visits to state i in one return to state j is given*

$$\mathbb{E}_j N_i(S_1(j)) = \frac{\pi_i}{\pi_j}.$$

Proof. Let $X_0 = j \in E$, then from renewal reward theorem for renewal sequence $S(i)$ and definition of π,

$$\lim_{n\in\mathbb{N}} P_j\{X_n = i\} = \lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} 1\{X_k = i\}}{n} = \frac{1}{\mathbb{E}_i[T_1(i)]} = \pi_i.$$

Result follows from rewriting of the above expression for renewal sequence $S(j)$ as

$$\lim_{n\in\mathbb{N}} P_j\{X_n = i\} = \lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} 1\{X_k = i\}}{n} = \frac{\mathbb{E}_j \sum_{k=1}^{S_1(j)} 1\{X_k = i\}}{\mathbb{E}_j S_1(j)} = \frac{\mathbb{E}_j N_i(S_1(j))}{\mathbb{E}_j T_1(j)} = \pi_j \mathbb{E}_j N_i(S_1(j)).$$

□

2.1 Ergodic theorem for Markov Chains

Proposition 2.6. *Let $\{X_n : n \in \mathbb{N}_0\}$ be an irreducible, aperiodic, and positive recurrent Markov chain on countable state space E with stationary distribution π. Let $f : E \to \mathbb{R}$, such that $\sum_{i\in E} |f(i)|\pi_i < \infty$, that is f is integrable over E with respect to π. Then for any initial distribution of X_0,*

$$\lim_{n\in\mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \sum_{i\in E} \pi_i f(i) \text{ almost surely.}$$

Proof. Fix $X_0 = i \in E$. Let $S(i)$ be sequence of successive instants at which state i is visited, with $S_0(i) = 0$. For all $p \geq 0$, let $R_{p+1} = \sum_{n=S_p(i)+1}^{S_{(p+1)}} f(X_n)$ be the net reward earned at the end of cycle $(p+1)$. Each cycle forms a renewal. By the strong Markov property, these cycles are independent. At each of these stopping times, Markov chain is in state $i \in E$. Since $\mathbb{E}_i S_1(i) = 1/\pi_i$, we get from renewal reward theorem

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} R_k}{n} = \pi_i \mathbb{E}_i[\sum_{n=1}^{S_1(i)} f(X_n)] = \pi_i \mathbb{E}_i \sum_{n=1}^{S_1(i)} \sum_{j\in E} f(j) 1\{X_n = j\}.$$

Using dominated convergence theorem, and substituting the mean number of visits to state j during successive return to state j, we can write

$$\lim_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} R_k}{n} = \pi_i \mathbb{E}_i \sum_{j\in E} f(j) \sum_{n=1}^{S_1(i)} 1\{X_n = j\} = \pi_i \sum_{j\in\mathbb{E}} f(j) \mathbb{E}_i N_j(S_1(i)) = \sum_{j\in\mathbb{E}} \pi_j f(j).$$

□