Lecture 15 : Continuous Time Markov Chains

1 Markov Process

A stochastic process $\{X(t) \in E,\ t \geqslant 0\}$ assuming values in a state space E and indexed by positive reals is a **Markov process** if the distribution of the future states conditioned on the present, is independent of the past. That is,

$$\Pr\{X(t+s) = j|X(u),\ u \in [0,s]\} = \Pr\{X(t+s) = j|X(s)\},\ \text{for all } s,t \geqslant 0 \text{ and } i,j \in E.$$

A Markov process with countable state space is referred to as **continuous time Markov chain (CTMC)**. We define the **transition probability** from state i at time s to state j at time $s+t$ as

$$P_{ij}(s,s+t) = \Pr\{X(s+t) = j|X(s) = i\}.$$

The Markov process has **homogeneous** transitions for all $i,j \in E, s,t \geqslant 0$, if

$$P_{ij}(s,s+t) = P_{ij}(0,t),$$

and we denote the **transition probability matrix** at time t by $P(t) = \{P_{ij}(t) = P_{ij}(0,t) : i,j \in E\}$. We will mainly be interested in continuous time Markov processes with homogeneous jump transition probabilities. A distribution π is an **equilibrium distribution** of a Markov process if

$$\pi P(t) = \pi,\ \ \forall t \geqslant 0.$$

1.1 Strong Markov property

For a process $\{X(t), t \geqslant 0\}$, if we denote the σ-algebra generated by realization of the process till time t as $\mathcal{F}_t = \sigma(\{X(u) : u \leqslant t\})$, then a random variable τ is a **stopping time** if for each $t \in \mathbb{R}_+$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

That is, a random variable τ is a stopping time if the event $\{\tau \leq t\}$ can be determined completely by the collection $\{X(u) : u \leqslant t\}$. A stochastic process X has **strong Markov property** if for any almost surely finite stopping time τ,

$$\Pr\{X(\tau+s) = j|X(u), u \leq \tau\} = \Pr\{X(\tau+s) = j|X(\tau)\}.$$

Lemma 1.1. *A homogeneous continuous time Markov chain X has the strong Markov property.*

Proof. Let τ be an almost surely finite stopping time with conditional distribution F on the collection of events $\{X(u) : u \leq s\}$. Then,

$$\Pr\{X(\tau+s) = j|X(u), u \leq \tau\} = \int_0^\infty dF(t)\Pr\{X(t+s) = j|X(u), u \leq t, \tau = t\} = \Pr\{X(\tau+s) = j|X(\tau)\}.$$

□

Since the CTMC is homogeneous and τ is almost surely finite stopping time, it is clear that

$$\Pr\{X(\tau+s) = j|X(\tau) = i\} = P_{ij}(\tau, \tau+s) = P_{ij}(0,s).$$

1.2 Jump and sojourn times

For any stochastic process on countable state space E that is right continuous with left limits (rcll), we wish to know following probabilities

$$\Pr\{X(s+t) = j | X(u),\ u \in [0,s]\},\ s,t \geq 0.$$

To this end, we define the sojourn time in any state, the jump times, and the jump transition probabilities. First, we define a stopping time for any stochastic process

$$\tau_t = \inf\{s > t : X(s) \neq X(t)\}.$$

For a homogeneous CTMC X, the distribution of $\tau_t - t$ only depends on $X(t)$ and doesn't depend on time t. Hence, we can define the following conditional distribution

$$F_i(u) \triangleq \Pr\{\tau_s - s \leq u | X(s) = i\}.$$

Lemma 1.2. *For a homogeneous CTMC X, there exists some $\nu_i > 0$, such that*

$$\Pr\{\tau_s - s > t | X(s) = i\} = e^{-t\nu_i}.$$

Proof. Using Markov property and the fact that τ_s is a stopping time, we can write

$$\begin{aligned}\bar{F}_i(u+v) &= \Pr\{\tau_s - s > u+v | X(s) = i, \mathcal{F}_s\} = \Pr\{\tau_s - s > u+v | \tau_s - s > u, X(s) = i, \mathcal{F}_s\} \Pr\{\tau_s - s > u | X(s) = i, \mathcal{F}_s\} \\ &= \Pr\{\tau_{s+u} - (s+u) > v | X(s+u) = i, \tau_s > u+s, \mathcal{F}_{s+u}\} \Pr\{\tau_s - s > u | X(s) = i, \mathcal{F}_s\} = \bar{F}_i(v)\bar{F}_i(u).\end{aligned}$$

The only continuous function $\bar{F}_i \in [0,1]$ that satisfies semigroup property is an exponential function with a negative exponent. $\square$

The **jump times** of a stochastic process $\{X(t), t \geqslant 0\}$ are defined as

$$S_0 = 0, \qquad S_n \triangleq \inf\{t > S_{n-1} : X(t) \neq X(S_{n-1})\}.$$

The **sojourn time** of this process staying in state $X(S_{n-1})$ is

$$T_n \triangleq (S_n - S_{n-1}).$$

Lemma 1.3. *Jump times $\{S_n : n \in \mathbb{N}\}$ are stopping times with respect to the process $\{X(t) : t \geqslant 0\}$.*

Proof. It is clear that $\{S_n \leq t\}$ is completely determined by the collection $\{X(u) : u \leqslant t\}$. $\square$

Lemma 1.4. *For a homogeneous CTMC, each sojourn time T_n is a continuous memoryless random variable, and the sequence of sojourn times $\{T_n : n \in \mathbb{N}\}$ are independent.*

Proof. We observe that $S_n = \tau_{S_{n-1}}$, and hence the conditional distribution of T_n given $\mathcal{F}_{S_{n-1}}$ is

$$\Pr\{T_n > y | \mathcal{F}_{S_{n-1}}\} = \exp(-y\nu_{X(S_{n-1})}) = \bar{F}_{X(S_{n-1})}(y),\ y \geq 0.$$

For independence of sojourn times, we show that the $(n+1)$th sojourn time is independent of nth jump time S_n. We can write the joint distribution as a conditional expectation

$$\Pr\{T_n > y, S_{n-1} \leq x | X(S_{n-1})\} = \mathbb{E}[\mathbb{E}[1\{\tau_{S_{n-1}} > S_{n-1} + y\} 1\{S_{n-1} \leq x\} | \mathcal{F}_{S_{n-1}}] | X(S_{n-1})].$$

Using strong Markov property of homogeneous CTMC X, we can write the right hand side as

$$\mathbb{E}[\bar{F}_{X(S_{n-1})} 1\{S_{n-1} \leq x\} | X(S_{n-1})] = \bar{F}_{X(S_{n-1})} \Pr\{S_{n-1} \leq x\}.$$

$\square$

Corollary 1.5. *If $X(S_n) = i$, then the random variable T_{n+1} has same distribution as the exponential random variable τ_i with rate ν_i.*

Inverse of mean sojourn time in state i is called the **transition rate** out of state i denote by ν_i. and typically $\nu_i < \infty$. If $\nu_i = \infty$, we call the state to be **instantaneous**.

1.3 Jump process

The **jump process** is a discrete time process $\{X^J(n) = X(S_n) : n \in \mathbb{N}_0\}$ derived from the continuous time stochastic process $\{X(t) : t \geqslant 0\}$ by sampling at jump times. The corresponding **jump transition probabilities** are defined by

$$p_{ij}(S_n) \triangleq \Pr\{X(S_n) = j | X(S_{n-1}) = i\},\ i, j \in E.$$

Lemma 1.6. *For any right continuous left limits stochastic process, the sum of jump transition probabilities $\sum_{j \neq i} p_{ij}(S_n) = 1$ for all $X(S_{n-1}) = i \in E$.*

Proof. It follows from law of total probability. □

Let $N(t)$ be the counting process associated with jump times sequence $\{S_n : n \in \mathbb{N}\}$. That is,

$$N(t) = \sum_{n \in \mathbb{N}} 1\{S_n \leq t\}.$$

Proposition 1.7. *For a homogeneous CTMC such that $\inf_{i \in E} \nu_i \geq \nu > 0$, then the jump times are almost surely finite stopping times.*

Proof. We obsevre that the jump times are sum of independent exponential random variables. Further by coupling, we can have a sequence of *iid* random variables $\{\overline{T}_n : n \in \mathbb{N}\}$, such that $T_n \leq \overline{T}_n$ and $\mathbb{E}\overline{T}_n = 1/\nu$ for each $n \in \mathbb{N}$. Hence, we have

$$S_n = \sum_{i=1}^{n} T_i \leq \sum_{i=1}^{n} \overline{T}_i \triangleq \overline{S}_n.$$

Result follows since $\overline{S}_n$ is the nth arrival instant of a Poisson process with rate ν. □

Lemma 1.8. *For a homogeneous CTMC, the jump probability from state $X(S_{n-1})$ to state $X(S_n)$ depend solely on $X(S_{n-1})$ and are independent of jump instants.*

Proof. We can re-write the jump probability as

$$\Pr\{X(S_n) = j | X(S_{n-1}) = i\} = P_{ij}(S_{n-1}, S_n) = P_{ij}(0, T_n).$$

For $T_n > x$, then we can write

$$\begin{aligned}\Pr\{T_n > x, X(S_n) = j | X(S_{n-1}) = i\} &= \Pr\{X(S_n) = j | T_n > x, X(S_{n-1}) = i\} \Pr\{T_n > x | X(S_{n-1}) = i\} \\ &= P_{ij}(S_{n-1} + x, S_n) \Pr\{\tau_{X(S_{n-1})} > S_{n-1} + x\} = P_{ij}(0, T_n - x)\bar{F}_i(x).\end{aligned}$$

Similarly, for $T_n \leq x$, we can write

$$\Pr\{T_n \leq x, X(S_n) = j | X(S_{n-1}) = i\} = \int_0^x \Pr\{X(S_n) = j | T_n = u, X(S_{n-1}) = i\} dF_i(u) = P_{ij}(0,0)F_i(x).$$

Hence for any $x \in \mathbb{R}_+$, we can write

$$P_{ij}(T_n) = P_{ij}(T_n - x)\bar{F}_i(x) + P_{ij}(0)F_i(x).$$

Result follows, since the only solution to this equation is $P_{ij}(T_n) = P_{ij}(0)$. □

Corollary 1.9. *For a homogeneous CTMC, the transition probabilities p_{ij} and sojourn times τ_i are independent.*

Corollary 1.10. *The jump process is a homogeneous Markov chain with countable state space E.*

1.4 Alternative construction of CTMC

Proposition 1.11. *A stochastic process $\{X(t) \in E, t \geqslant 0\}$ is a CTMC iff*

a. sojourn times are independent and exponentially distributed with rate ν_i where $X(S_{n-1}) = i$, and

b. jump transition probabilities $p_{ij}(S_n)$ are independent of jump times S_n, such that $\sum_{i \neq j} p_{ij} = 1$.

Proof. Necessity of two conditions follows from Lemma 1.4 and 1.8. For sufficiency, we assume both conditions and show that Markov property holds and the transition probability is homogeneous. Using jump time independence of jump probabilities, we can write

$$\Pr\{X(t+s) = j | X(s) = i, \mathcal{F}_s\} = \sum_{k=0}^{\infty} \Pr\{X(t+s) = j, N(s+t) - N(s) = k | N(s), X(s) = i, \mathcal{F}_s\}.$$

Since sojourn times are memoryless and depend only on the previous state, it follows that $N(t+s) - N(s)$ is independent of $N(s)$ and

$$\Pr\{X(t+s) = j, N(s+t) - N(s) = k | N(s), X(s) = i, \mathcal{F}_s\} = \Pr\{X(t+s) = j, N(s+t) - N(s) = k | X(s) = i\}.$$

This shows that the X is a Markov process.

The distribution of $N(s+t) - N(s)$ conditioned on $X(s) = i$ has the same distribution as $N(t)$ conditioned on $X(0) = i$. Further, the jump probabilities are independent of jump times. Together, it follows that

$$\Pr\{X(t+s) = j, N(s+t) - N(s) = k | X(s) = i\} = \Pr\{X(t) = j, N(t) - N(0) = k | X(0) = i\}.$$

Therefore, it follows that X is a homogeneous Markov process. □

A Exponential random variables

Lemma A.1. *Let X be an exponential random variable, and S be any positive random variable, independent of X. Then,*

$$\Pr\{X > S + u | X > S\} = \Pr\{X > u\}.$$

Proof. Let the distribution of S be F. Then, using memoryless property of exponential random variable, we can write

$$\Pr\{X > S + u | X > S\} = \int_0^\infty dF(v) \Pr\{X > u + v | X > v\} = \Pr\{X > u\} \int_0^\infty dF(v) = \Pr\{X > u\}.$$

□