Lecture 16 : Evolution of Markov Processes

1 Regularity and Stationarity

A CTMC is called **regular** if for all finite $t \in \mathbb{R}_+$, number of jumps $N(t)$ is almost surely finite. That is, for all $t \in \mathbb{R}_+$

$$\Pr\{N(t) < \infty\} = 1.$$

Lemma 1.1. *A homogeneous CTMC is regular if* $\sup_{i \in E} \nu_i < \nu < \infty$.

Proof. By coupling, we can have a sequence of *iid* random variables $\{\underline{T}_n : n \in \mathbb{N}\}$, such that $\underline{T}_n \leq T_n$ and $\mathbb{E}X_n = \nu$ for each $n \in \mathbb{N}$. Let $\underline{m}(t)$ be the associated renewal function with the sequence $\underline{T}$, then we can write

$$\Pr\{N(t) < \infty\} = \sum_{n \in \mathbb{N}_0} \Pr\{S_n \geq t\} = 1 + m(t) \leq 1 + \underline{m}(t).$$

□

Consider the following example of a non-regular CTMC, where for all $i \in \mathbb{N}$

$$p_{i,i+1} = 1, \nu_i = i^2.$$

Clearly, $\sup_{i \in E} \nu_i = \infty$, and hence it is not regular.

1.1 Properties of transition matrix

For each t, we have the transition matrix $P(t)$.

Lemma 1.2 (continuity). *Transition matrix $P(t)$ for a homogeneous CTMC X is a continuous function of time $t \in \mathbb{R}_+$, such that*

$$\lim_{t \downarrow 0} P(t) = I.$$

Proof. It follows from continuity of probability functions and alternate characterization of homogeneous CTMC. □

Lemma 1.3 (semigroup property). *Transition matrix $P(t)$ satisfies the semigroup property*

$$P(s+t) = P(s)P(t).$$

Since each entry of $P(t)$ is a probability, this leads to characterization of $P(t)$ completely.

Proof. From homogeneity of Markov chains, we can write the (i,j)th entry of $P(s+t)$ as

$$P_{ij}(0,s+t) = \sum_{k\in E} P_{ik}(0,s)P_{kj}(s,s+t) = \sum_{k\in E} P_{ik}(0,s)P_{kj}(0,t) = [P(s)P(t)]_{ij}.$$

□

For a homogeneous CTMC with transition matrix $P(t)$, the **generator matrix** $Q \in \mathbb{R}^{E\times E}$ is defined as the following limit when it exists

$$Q \triangleq \lim_{t\downarrow 0} \frac{P(t)-I}{t}.$$

Theorem 1.4. *For a homogeneous CTMC, the generator matrix exists and is defined in terms of sojourn time rates $\{\nu_i : i \in E\}$, and jump transition matrix $p = \{p_{ij} : i,j \in E\}$ as*

$$Q_{ii} = -\nu_i, \qquad Q_{ij} = \nu_i p_{ij}.$$

Proof. We can expand the (i,j)th entry of transition matrix in terms of disjoint events $\{N(t) = n\}$ as

$$P_{ij}(t) = \Pr\{X(t) = j|X(0) = i\} = \sum_{n\in\mathbb{N}_0} \Pr\{X(t) = j, N(t) = n|X(0) = i\}.$$

We can write the upper and lower bound as

$$\sum_{n=0}^{1} \Pr\{X(t) = j, N(t) = n|X(0) = i\} \le P_{ij}(t) \le \sum_{n=0}^{1} \Pr\{X(t) = j, N(t) = n|X(0) = i\} + \Pr\{N(t) \ge 2\}.$$

For $t > 0$, we can compute for $j \neq i \in E$

$$\Pr\{X(t) = i, N(t) = 0|X(0) = i\} = e^{-\nu_i t}, \quad \Pr\{X(t) = i, N(t) = 1|X(0) = i\} = 0,$$
$$\Pr\{X(t) = j, N(t) = 0|X(0) = i\} = 0, \quad \Pr\{X(t) = j, N(t) = 1|X(0) = i\} = p_{ij}\int_0^t \nu_i e^{-\nu_j(t-u)} e^{-\nu_i u} du.$$

Since $\{N(t) \ge 2\}$ is of order $o(t)$ for small t, we can write

$$\frac{P_{ij}(t) - I_{ij}}{t} = -\left(\frac{1-e^{-\nu_i t}}{t}\right) I_{ij} + \nu_i p_{ij} \frac{(e^{-\nu_j} - e^{-\nu_i t})}{(\nu_i - \nu_j)t}(1 - I_{ij}) + o(t).$$

Taking limit as $t \downarrow 0$, we get the result. □

Corollary 1.5. *For each state $i \in E$, the generator matrix $Q \in \mathbb{R}^{E\times E}$ for a homogeneous CTMC satisfies*

$$0 \le -Q_{ii} < \infty, \qquad Q_{ij} \ge 0, \qquad \sum_{j\in E} Q_{ij} = 0.$$

1.2 Chapman Kolmogorov equations

Theorem 1.6 (backward equation). *For a homogeneous CTMC with transition matrix $P(t)$ and generator matrix Q, we have*

$$\frac{dP(t)}{dt} = QP(t), \ \ t \geqslant 0.$$

Proof. Using semigroup property of transition probability matrix $P(t)$ for a homogeneous CTMC, we can write

$$\frac{P(t+h)-P(t)}{h} = \frac{(P(h)-I)}{h}P(t).$$

Taking limits $h \downarrow 0$ and exchanging limits and summation, we get

$$\frac{dP_{ij}(t)}{dt} = \sum_{k\neq i} Q_{ik}P_{kj}(t) - \nu_i P_{ij}(t).$$

Now the exchange of limit and summation has to be justified. For any finite subset $F \subset E$, we have

$$\liminf_{h\downarrow 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \sum_{k\in F\setminus\{i\}} \liminf_{h\downarrow 0} \frac{P_{ik}(h)}{h} P_{kj}(t) = \sum_{k\in F\setminus\{i\}} Q_{ik}P_{kj}(t).$$

Since, above is true for any finite set $F \subset E$, taking supremum over increasing sets F, we get the lower bound. For the upper bound, we observe for any finite subset $F \subseteq E$

$$\begin{aligned}
\limsup_{h\downarrow 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) &\leq \limsup_{h\downarrow 0} \left(\sum_{k\in F\setminus\{i\}} \frac{P_{ik}(h)}{h} P_{kj}(t) + \sum_{k\notin F\setminus\{i\}} \frac{P_{ik}(h)}{h} \right) \\
&= \limsup_{h\downarrow 0} \left(\sum_{k\in F\setminus\{i\}} \frac{P_{ik}(h)}{h} P_{kj}(t) + \frac{1-P_{ii}(h)}{h} - \sum_{k\in F\setminus\{i\}} \frac{P_{ik}(h)}{h} \right) \\
&= \sum_{k\in F\setminus\{i\}} Q_{ik}P_{kj}(t) + \nu_i - \sum_{k\in F\setminus\{i\}} Q_{ik}.
\end{aligned}$$

□

Theorem 1.7 (forward equation). *For a homogeneous CTMC with transition matrix $P(t)$ and generator matrix Q, we have*

$$\frac{dP(t)}{dt} = P(t)Q.$$

Proof. Using semigroup property of transition probability matrix $P(t)$ for a homogeneous CTMC, we can write

$$\frac{P(t+h)-P(t)}{h} = P(t)\frac{(P(h)-I)}{h}.$$

Taking limits $h \downarrow 0$, we get

$$\frac{dP_{ij}(t)}{dt} = \sum_{k\neq j} P_{ik}(t)Q_{kj} - \nu_j P_{ij}(t).$$

By taking limiting value for increasing sequence of finite sets $F \subseteq E$, we obtain the lower bound

$$\sum_{k\neq j} P_{ik}(t)Q_{kj} \leq \liminf_{h\downarrow 0} \sum_{k\neq j} P_{ik}(t)\frac{P_{kj}(h)}{h}.$$

To obtain the upper bound, we observe for any finite subset $F \subseteq E$,

$$\limsup_{h\downarrow 0} \sum_{k\neq i} P_{ik}(t)\frac{P_{kj}(h)}{h} \leq \limsup_{h\downarrow 0} \left(\sum_{k\in F\setminus\{j\}} P_{ik}(t)\frac{P_{kj}(h)}{h} + \frac{1-P_{jj}(h)}{h} - \sum_{k\in F\setminus\{j\}} \frac{P_{kj}(h)}{h} \right).$$

□

Corollary 1.8. *For a homogeneous CTMC with finite state space E, the transition matrix P(t) and generator matrix Q, we have*

$$P(t) = e^{tQ} = I + \sum_{n \in \mathbb{N}} \frac{t^n Q^n}{n!}, \; t \geqslant 0.$$

Corollary 1.9. *For a homogeneous CTMC with finite state space E, the transition matrix P(t) and generator matrix Q, the stationary distribution satisfies*

$$\pi Q = 0, \qquad \pi_i > 0, \qquad \sum_{i \in E} \pi_i = 1.$$

Proof. We can define probability of being in state $j \in E$ at time t by

$$\pi_j(t) = \Pr\{X(t) = j\}.$$

By Markov property, we have $\pi(t) = \{\pi_j(t) : j \in E\}$ by

$$\pi(t) = \pi(0)P(t).$$

For a stationary distribution, we have $\pi = \pi P(t)$ and taking derivatives on both sides, we get the result. □

1.3 Transition graph

The directed transition graph consists of vertex set E and the edges being

$$\{(i, j) : p_{ij} > 0, i \neq j\}.$$

The weights of the directed edges are given by $w_{ij} = \nu_i p_{ij}$.

A Generator matrix

A **generator matrix** denoted by $Q \in \mathbb{R}^{E \times E}$ is defined in terms of sojourn times $\{\nu_i, i \in E\}$ and jump transition probabilities $\{p_{ij}, i \neq j \in E\}$ of a CTMC as

i_ $q_{ii} = -\nu_i$,

ii_ $q_{ij} = \nu_i p_{ij}$.

Lemma A.1. *A matrix Q is a generator matrix for a CTMC iff for each $i \in I$,*

*i_* $0 \leq -q_{ii} < \infty$,

*ii_* $q_{ij} \geq 0$,

*iii_* $\sum_{j \in I} q_{ij} = 0$.

From the Q matrix, we can construct the whole CTMC. In DTMC, we had the result $P^{(n)}(i, j) = (P^n)_{i,j}$. We can generalize this notion in the case of CTMC as follows: $P = e^Q \triangleq \sum_{k \in \mathbb{N}_0} \frac{Q^k}{k!}$. Observe that $e^{Q_1 + Q_2} = e^{Q_1} e^{Q_2}$, $e^{nQ} = (e^Q)^n = P^n$.

Theorem A.2. *Let Q be a finite sized matrix. Let $P(t) = e^{tQ}$. Then $\{P(t),\ t \geq 0\}$ has the following properties:*

1. *$P(s+t) = P(s)P(t),\ \forall s,\ t$ (semi group property).*
2. *$P(t),\ t \geq 0$ is the unique solution to the forward equation, $\frac{dP(t)}{dt} = P(t)Q,\ P(0) = I$.*
3. *And the backward equation $\frac{dP(t)}{dt} = QP(t),\ P(0) = I$.*
4. *For all $k \in \mathbb{N}$, $\frac{d^kP(t)}{d^k(t)}|_{t=0} = Q^k$.*

Proof. $\frac{dM(t)e^{-tQ}}{dt} = 0$, $M(t)e^{-tQ}$ is constant. $M(t)$ is any matrix satisfying the forward equation. □

Theorem A.3. *A finite matrix Q is a generator matrix for a CTMC iff $P(t) = e^{tQ}$ is a stochastic matrix for all $t \geq 0$.*

Proof. $P(t) = I + tQ + O(t^2)$ $(f(t) = O(t) \Rightarrow \frac{f(t)}{t} \leq c$, for small $t,\ c < \infty$). $q_{ij} \geq 0$ if and only if $P_{ij}(t) \geq 0,\ \forall i \neq j$ and $t \geq 0$ sufficiently small. $P(t) = P(\frac{t}{n})^n$. Note that if Q has zero row sums, Q^n also has zero row sums.

$$\sum_j [Q^n]_{ij} = \sum_j \sum_k [Q^{n-1}]_{ik} Q_{kj} = \sum_j \sum_k Q_{kj} [Q^{n-1}]_{ik} = 0.$$
$$\sum_j P_{ij}(t) = 1 + \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \sum_j [Q^n]_{ij} = 1 + 0 = 1.$$

Conversely $\sum_j P_{ij}(t) = 1,\ \forall t \geq 0$, then $\sum_j Q_{ij} = \frac{dP_{ij}(t)}{dt} = 0$. □