Lecture 18: Reversibility

1 Introduction

A stochastic process $\{X(t) \in I : t \in T\}$ is **reversible** if $(X(t_i) : i \in [n])$ has the same distribution as $(X(\tau - t_i) : i \in [n])$ for all $t_i, \tau \in T, i \in [n]$.

Lemma 1.1. *A reversible process is stationary.*

Proof. Since $X(t)$ is reversible, both $(X(t_i) : i \in [n])$ and $(X(\tau + t_i) : i \in [n])$ have the same distribution as $(X(-t_i) : i \in [n])$. □

Theorem 1.2. *A stationary Markov chain with state space I and probability transition matrix P is reversible iff there exists a probability distribution π, that satisfy the detailed balanced conditions*

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall i, j \in I. \tag{1}$$

When such a distribution π exists, it is the equilibrium distribution of the process.

Proof. We assume that $X(t)$ is reversible, and hence stationary. We denote the stationary distribution by π, and by reversibility of $X(t)$ we have

$$\Pr\{X(t) = i, X(t+1) = j\} = \Pr\{X(t) = j, X(t+1) = i\},$$

and hence we obtain the detailed balanced conditions (1).

Conversely, let π be the distribution that satisfies the detailed balanced conditions, then summing up both sides over $j \in I$, we see that this distribution is the equilibrium distribution. Let $j_i \in I$ for $i \in [m]$, and we write

$$\Pr\{X(t+i-1) = j_{i-1}, i \in [m]\} = \pi(j_0) \prod_{i=1}^{m} P(j_{i-1}, j_i),$$

$$\Pr\{X(t'+i-1) = j_{m-i+1}, i \in [m]\} = \pi(i_m) \prod_{j=m}^{1} P(j_i, j_{i-1}).$$

From detailed balanced equations (1) it follows that RHS of above two equations are identical. Taking $\tau = t + t' + m$, we deduce that $X(t)$ is reversible. □

Theorem 1.3. *A stationary Markov process with state space I and generator matrix Q is reversible iff there exists a probability distribution π, that satisfy the detailed balanced conditions*

$$\pi_i Q_{ij} = \pi_j Q_{ji}, \quad \forall i, j \in I. \tag{2}$$

When such a distribution π exists, it is the equilibrium distribution of the process.

Proof. We assume that $X(t)$ is reversible, and hence stationary. We denote the stationary distribution by π, and by reversibility of $X(t)$ we have

$$\Pr\{X(t) = i, X(t+\tau) = j\} = \Pr\{X(t) = j, X(t+\tau) = i\},$$

and hence we obtain the detailed balanced conditions (2) by taking limit $\tau \to 0$.

Conversely, let π be the distribution that satisfies the detailed balanced conditions, then summing up both sides over $j \in I$, we see that this distribution is the equilibrium distribution. Consider now the behavior of stationary process $X(t)$ in $[-T, T]$. Process may start at time $-T$ in state j_1 and sees m states by time T. For $i \in [m-1]$, we can define

$$S_1 = -T, \qquad S_{i+1} = \inf\{t > S_i : X(t) \neq X(S_i)\}, \qquad S_{m+1} = T.$$

That is, the process spends period $S_{i+1} - S_i$ in state j_i for $i \in [m]$, and transitions to state j_{i+1} at instant S_{i+1} for $i \in [m-1]$. Probability of this event is

$$\Pr\{X(t) = j_i,\ t \in [S_i, S_{i+1}), i \in [m]\} = \pi(j_1) \prod_{i=1}^{m-1} Q(j_i, j_{i+1}) \prod_{i=1}^{m} e^{-\nu(j_i)(S_{i+1}-S_i)}.$$

Consider the stationary process that start in state j_m at time $\tau - T$ such that, for $i \in [m]$

$$X(t) = j_i,\ t \in [\tau - S_{i+1}, \tau - S_i).$$

Probability of this event is

$$\Pr\{X(t) = j_i,\ t \in [\tau - S_{i+1}, \tau - S_i), i \in [m]\} = \pi(j_m) \prod_{i=2}^{m} Q(j_i, j_{i-1}) \prod_{i=1}^{m} e^{-\nu(j_i)(S_{i+1}-S_i)}.$$

From detailed balance equation (2) it follows that

$$\pi(j_1) \prod_{i=1}^{m-1} Q(j_i, j_{i+1}) = \pi(j_m) \prod_{i=2}^{m} Q(j_i, j_{i-1}).$$

Hence, it follows that $X(t)$ is reversible. □

The **probability flux** from state i to state j is defined as $\pi_i Q_{ij}$.

Lemma 1.4. *For a stationary Markov process, probability flux balances across a cut $A \subseteq I$, that is*

$$\sum_{i \in A} \sum_{j \notin A} \pi_i Q_{ij} = \sum_{i \in A} \sum_{j \notin A} \pi_j Q_{ji}.$$

Proof. From full balance condition $\pi Q = 0$, we get

$$\sum_{j \in A} \sum_{i \in I} \pi_i Q_{ij} = \sum_{j \in A} \sum_{i \in I} \pi_j Q_{ji} = 0.$$

Further, we have the following identity

$$\sum_{j \in A} \sum_{i \in A} \pi_i Q_{ij} = \sum_{j \in A} \sum_{i \in A} \pi_j Q_{ji}.$$

Subtracting the second identity from the first, we get the result. □

Corollary 1.5. *For $A = \{i\}$, the above equation reduces to the full balance equation for state i, i.e.,*

$$\begin{aligned} \sum_{i \neq j} \pi_i Q_{ij} &= \sum_{j \neq i} \pi_j Q_{ji} \\ \Rightarrow \sum_{j \in I} \pi_i Q_{ij} - \pi_i Q_{ii} &= \sum_{j \in I} \pi_j Q_{ji} - \pi_i Q_{ii} \\ \Rightarrow 0 &= \sum_{j \in I} \pi_j Q_{ji}. \end{aligned}$$

Is every Markov chain reversible? If we try to prove the equations necessary for time reversibility, $x_i P_{ij} = x_j P_{ji}$ for all $i, j \in I$, for any arbitrary Markov chain, one may not end up getting any solution. This is so because, if $P_{ij} P_{jk} > 0$, then

$$\frac{x_i}{x_k} = \frac{P_{kj} P_{ji}}{P_{ij} P_{jk}} \neq \frac{P_{ki}}{P_{ik}}.$$

Thus, we see that a necessary condition for time reversibility is

$$P_{ij} P_{jk} P_{ki} = P_{ik} P_{kj} P_{ji} \text{ for all } i, j, k \in I.$$

Theorem 1.6 (Kolmogorov's criterion for reversibility of Markov chains). *A stationary Markov chain is time reversible if and only if starting in state i, any path back to state i has the same probability as the reversed path, for all initial states $i \in I$. That is, if*

$$P_{ii_1} P_{i_1 i_2} \dots P_{i_k i} = P_{ii_k} P_{i_k i_{k-1}} \dots P_{i_1 i}.$$

Proof. The proof of necessity is as indicated above. To see the sufficiency part, fix states i, j. For any positive integer k, we compute

$$(P^k)_{ij} P_{ji} = \sum_{i_1, i_2, \dots i_k} P_{ii_1} \dots P_{i_k j} P_{ji} = \sum_{i_1, i_2, \dots i_k} P_{ij} P_{ji_k} \dots P_{i_1 i} = P_{ij} (P^k)_{ji}.$$

Taking $k \to \infty$ and noticing that $(P^k)_{ij} \overset{k \to \infty}{\to} \pi_j \; \forall i, j \in I$, we get the desired result by appealing to Theorem 1.2. □