Lecture 23 : Martingales

1 Martingales

A **filtration** is an increasing sequence of σ-fields, with nth σ-field denoted by $\mathcal{F}_n$. A sequence $X = \{X_n : n \in \mathbb{N}\}$ of random variables is said to be **adapted** to the filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if $X_n \in \mathcal{F}_n$. A discrete stochastic process $\{X_n,\ n \in \mathbb{N}\}$ is said to be a **martingale** with respect to $\{\mathcal{F}_n : n \in \mathbb{N}\}$ if

i_ $\mathbb{E}[|X_n|] < \infty$,

ii_ X_n is adapted to $\mathcal{F}_n$,

iii_ $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$, for each $n \in \mathbb{N}$.

If the equality in third condition is replaced by $\leq$ or $\geq$, then the process is called **supermartingale** or **submartingale**, respectively. For a discrete stochastic process $X = \{X_n : n \in \mathbb{N}\}$, its **natural filtration** is defined as

$$\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n).$$

Corollary 1.1. *For a martingale X adapted to a filtration $\mathcal{F}$, for each $n \in \mathbb{N}$*

$$\mathbb{E}X_n = \mathbb{E}X_1.$$

(Simple random walk). *Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of independent random variables with mean $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i| < \infty$ for each $i \in \mathbb{N}$. Let $Z_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ for each $n \in \mathbb{N}$. Then, $\{Z_n,\ n \in \mathbb{N}\}$ is a martingale with respect to the natural filtration of X. This follows, since $\mathbb{E}Z_n = 0$ and*

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_n + X_{n+1}|\mathcal{F}_n] = Z_n.$$

(Product martingale). *Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of independent random variables with mean $\mathbb{E}X_i = 1$ and $\mathbb{E}|X_i| < \infty$ for each $i \in \mathbb{N}$. Let $Z_n = \Pi_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then, $\{Z_n,\ n \in \mathbb{N}\}$ is a martingale with respect to the natural filtration of X. This follows, since $\mathbb{E}Z_n = 1$ and*

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_n X_{n+1}|\mathcal{F}_n] = Z_n.$$

(Branching process). *Consider a population where each individual i can produce an independent random number of offsprings Z_i in its lifetime, given by a common distribution $P = \{P_j : j \in \mathbb{N}_0\}$ and mean $\mu = \sum_{j\in\mathbb{N}} jP_j$. Let X_n denote the size of the nth generation, which is same as number of offsprings generated by $(n-1)$th generation. The discrete stochastic process $\{X_n \in \mathbb{N}_0 : n \in \mathbb{N}\}$ is called a branching process. Let $X_0 = 1$ and $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$, then,*

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i.$$

Conditioning on X_{n-1} yields, $\mathbb{E}[X_n] = \mu^n$ where μ is the mean number of offspring per individual. Then $\{Y_n = X_n/\mu^n : n \in \mathbb{N}\}$ is a martingale because $\mathbb{E}[Y_n] = 1$ and

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \frac{1}{\mu^{n+1}}\mathbb{E}[\sum_{i=1}^{X_n} Z_i|\mathcal{F}_n] = \frac{X_n}{\mu^n} = Y_n.$$

(Doob's Martingale). *Let X be an arbitrary random variable such that $\mathbb{E}[|X|] < \infty$, and $Y = \{Y_n : n \in \mathbb{N}\}$ be an arbitrary random sequence. Let $\mathcal{F}$ be the natural filtration associated with the stochastic process Y, then*

$$X_n = \mathbb{E}[X|\mathcal{F}_n]$$

is a martingale. The integrability condition can be directly verified, and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_n] = X_n.$$

(Centralized Doob sequence). *For any sequence of random variables $X = \{X_n : n \in \mathbb{N}\}$ and its natural filtration $\mathcal{F}$, the random variables $X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}]$ have zero mean, then*

$$Z_n = \sum_{i=1}^{n}(X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}])$$

is a martingale with respect to $\mathcal{F}$, provided $\mathbb{E}|Z_n| < \infty$. To verify the same,

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_n + X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}]|\mathcal{F}_n] = Z_n + \mathbb{E}[X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}]] = Z_n.$$

Lemma 1.2. *If $X = \{X_n : n \in \mathbb{N}\}$ is a martingale with respect to a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ and f is a convex function, then $\{f(X_n) : n \in \mathbb{N}\}$ is a sub martigale with respect to the same filtration.*

Proof. The result is a direct consequence of Jensen's inequality.

$$\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] \geq f(\mathbb{E}[X_{n+1}|\mathcal{F}_n]) = f(X_n).$$

□

Corollary 1.3. *Let $a \in \mathbb{R}$ be a constant.*

i_ If $\{X_n : n \in \mathbb{N}\}$ is a submartingale, then so is $\{(X_n - a)_+ : n \in \mathbb{N}\}$.

ii_ If $\{X_n : n \in \mathbb{N}\}$ is a supermartingale, then so is $\{X_n \wedge a : n \in \mathbb{N}\}$.

1.1 Stopping Times

Consider a discrete filtration $\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{N}_0\}$. A positive integer valued, possibly infinite, random variable N is said to be a **random time** with respect to the filtration $\mathcal{F}$, if the event $\{N = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. If $\Pr\{N < \infty\} = 1$, then the random time N is said to be a **stopping time**. A sequence $\{H_n : n \in \mathbb{N}\}$ is **predictable** with respect to the the filtration $\mathcal{F}$, if $H_n \in \mathcal{F}_{n-1}$ for each $n \in \mathbb{N}$. Further, we define

$$(H \cdot X)_n \triangleq \sum_{m=1}^{n} H_m (X_m - X_{m-1}).$$

Theorem 1.4. *Let $\{X_n : n \in \mathbb{N}_0\}$ be a super martingale with respect to a filtration $\mathcal{F}$. If $H = \{H_n : n \in \mathbb{N}\}$ is predictable with respect to $\mathcal{F}$ and each H_n is non-negative and bounded, then $(H \cdot X)_n$ is a super martingale w.r.t. $\mathcal{F}$.*

Proof. It follows from the definition,

$$\mathbb{E}[(H \cdot X)_{n+1} | \mathcal{F}_n] = \mathbb{E}[H_{n+1}(X_{n+1} - X_n) + (H \cdot X)_n | \mathcal{F}_n] = H_{n+1}(\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n) + (H \cdot X)_n \leq (H \cdot X)_n.$$

□

Lemma 1.5. *If $\{X_i : i \in \mathbb{N}\}$ is a submartingale and T is a stopping time such that* $\Pr\{T \leq n\} = 1$ *then*

$$\mathbb{E}X_1 \leq \mathbb{E}X_T \leq \mathbb{E}X_n.$$

Proof. Since T is bounded, it follows from Martingale stopping theorem, that $\mathbb{E}X_T \geq \mathbb{E}X_1$. Now, since T is a stopping time, we see that for $\{T = k\}$

$$\mathbb{E}[X_n 1\{T = k\} | \mathcal{F}_T, T = k] = \mathbb{E}[X_n 1\{T = k\} | \mathcal{F}_k] \geq X_k 1\{T = k\} = X_T 1\{T = k\}.$$

Result follows by taking expectation on both sides and summing over k. That is,

$$\mathbb{E}X_n = \mathbb{E} \sum_{k=1}^{n} X_n 1\{T = k\} \geq \mathbb{E} \sum_{k=1}^{n} X_T 1\{T = k\} = \mathbb{E}X_T.$$

□

Corollary 1.6. *Let T be a stopping time and $\{X_n : n \in \mathbb{N}\}$ be a supermartingale, then $\{X_{T \wedge n} : n \in \mathbb{N}\}$ os a supermartingale.*

1.2 Stopped process

Consider a discrete stochastic process $X = \{X_n : n \in \mathbb{N}\}$ adapted to a discrete filtration $\mathcal{F}$. Let T be a random time for the filtration $\mathcal{F}$, then the **stopped process** $\{X_{T\wedge n} : n \in \mathbb{N}\}$ is defined as

$$X_{T\wedge n} = X_n 1_{\{n\leq T\}} + X_T 1_{\{n>T\}}.$$

Proposition 1.7. *Let $\{X_n : n \in \mathbb{N}\}$ be a martingale with a discrete filtration $\mathcal{F}$. If T is an integer random time for the filtration $\mathcal{F}$, then the stopped process $\{X_{T\wedge n}\}$ is a martingale.*

Proof. We observe that $H = \{1\{n \leq T\} : n \in \mathbb{N}\}$ is a non-negative, predictable, and bounded sequence, since

$$\{n \leq T\} = \{T > n-1\} = \{T \leq n-1\}^c = (\cup_{i=0}^{n-1}\{T = i\})^c = \cap_{i=1}^{n-1}\{T \neq i\} \in \mathcal{F}_{n-1}.$$

In terms of the predictable and bounded sequence H, we can write the stopped process as

$$X_{T\wedge n} = X_0 + \sum_{m=1}^{T\wedge n}(X_m - X_{m-1}) = X_0 + \sum_{m=1}^{n} 1\{m \leq T\}(X_m - X_{m-1}) = X_0 + (H\cdot X)_n.$$

Therefore from the previous theorem we have

$$\mathbb{E}X_{T\wedge n} = \mathbb{E}X_{T\wedge 1} = \mathbb{E}X_1.$$

□

Remark 1.8. For any martingale $\{X_n : n \in \mathbb{N}\}$ w.r.t. $\mathcal{F}$, we have $\mathbb{E}X_{T\wedge n} = \mathbb{E}X_1$, for all n. Now assume that T is a stopping time w.r.t. $\mathcal{F}$. It is immediate that stopped process converges almost surely to X_T, i.e.

$$\Pr\left\{\lim_{n\in\mathbb{N}} X_{T\wedge n} = X_T\right\} = 1.$$

We are interested in knowing under what conditions will we have convergence in mean.

Theorem 1.9 (Martingale stopping theorem). *Let X be a martingale and T be a stopping time adapted to a discrete filtration $\mathcal{F}$. Then, the random variable X_T is integrable and the stopped process $X_{T\wedge n}$ converges in mean to X_T, i.e.*

$$\lim_{n\in\mathbb{N}} \mathbb{E}X_{T\wedge n} = \mathbb{E}X_T = \mathbb{E}X_1,$$

if either of the following conditions holds true.

(i) T is bounded,

(ii) $X_{T\wedge n}$ is uniformly bounded,

(iii) $\mathbb{E}T < \infty$, and for some real positive K, we have $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_{n+1} - X_n||\mathcal{F}_n] < K$.

Proof. We show this is true for all three cases.

(i) Let K be the bound on T then for all $n \geq K$, we have $X_{T\wedge n} = X_T$, and hence it follows that

$$\mathbb{E}X_1 = \mathbb{E}X_{T\wedge n} = \mathbb{E}X_T,\ \forall n \geq K.$$

(ii) Dominated convergence theorem implies the result.

(iii) Since T is integrable and

$$X_{T\wedge n} \leq |X_1| + KT,$$

we observe that $X_{T\wedge n}$ is bounded by an integrable random variable, and hence result follows from dominated convergence theorem.

$\square$

Corollary 1.10 (Wald's Equation). *If T is a stopping time for $\{X_i : i \in \mathbb{N}\}$ iid with $\mathbb{E}|X| < \infty$ and $\mathbb{E}T < \infty$, then*

$$\mathbb{E}\sum_{i=1}^{T} X_i = \mathbb{E}T\mathbb{E}X.$$

Proof. Let $\mu = \mathbb{E}X$. Then $\{Z_n = \sum_{i=1}^{n}(X_i - \mu) : n \in \mathbb{N}\}$ is a martingale adapted to natural filtration for X, and hence from the Martingale stopping theorem, we have $\mathbb{E}Z_T = \mathbb{E}Z_1 = 0$. But

$$\mathbb{E}[Z_T] = \mathbb{E}\sum_{i=1}^{T} X_i - \mu\mathbb{E}T.$$

Observe that condition (iii) for Martingale stopping theorem to hold can be directly verified. Hence the result follows. $\square$