# Lecture 24: Martingale Convergence Theorem

## 1 Martingale Convergence Theorem

Before we state and prove martingale convergence theorem, we state some results which will be used in the proof of the theorem.

**Lemma 1.1.** *If* $\{X_i : i \in \mathbb{N}\}$ *is a submartingale and* $T$ *is a stopping time such that* $\Pr\{T \le n\} = 1$ *then*

$$\mathbb{E}X_1 \le \mathbb{E}X_T \le \mathbb{E}X_n.$$

*Proof.* Since $T$ is bounded, it follows from Martingale stopping theorem, that $\mathbb{E}X_T \ge \mathbb{E}X_1$. Now, since $T$ is a stopping time, we see that for $\{T = k\}$

$$\mathbb{E}[X_n 1\{T = k\} | \mathcal{F}_T, T = k] = \mathbb{E}[X_n 1\{T = k\} | \mathcal{F}_k] \ge X_k 1\{T = k\} = X_T 1\{T = k\}.$$

Result follows by taking expectation on both sides and summing over $k$. That is,

$$\mathbb{E}X_n = \mathbb{E}\sum_{k=1}^{n} X_n 1\{T = k\} \ge \mathbb{E}\sum_{k=1}^{n} X_T 1\{T = k\} = \mathbb{E}X_T.$$

□

**Lemma 1.2.** *If* $X = \{X_n : n \in \mathbb{N}\}$ *is a martingale with respect to a filtration* $\{\mathcal{F}_n : n \in \mathbb{N}\}$ *and* $f$ *is a convex function, then* $\{f(X_n) : n \in \mathbb{N}\}$ *is a sub martigale with respect to the same filtration.*

*Proof.* The result is a direct consequence of Jensen's inequality.

$$\mathbb{E}[f(X_{n+1}) | \mathcal{F}_n] \ge f(\mathbb{E}[X_{n+1} | \mathcal{F}_n]) = f(X_n).$$

□

**Corollary 1.3.** *Let* $a \in \mathbb{R}$ *be a constant.*

*i_ If* $\{X_n : n \in \mathbb{N}\}$ *is a submartingale, then so is* $\{(X_n - a)_+ : n \in \mathbb{N}\}$.

*ii_ If* $\{X_n : n \in \mathbb{N}\}$ *is a supermartingale, then so is* $\{X_n \wedge a : n \in \mathbb{N}\}$.

Let $X = \{X_n : n \in \mathbb{N}_0\}$ be a submartingale. Let $a < b$ and $N_0 = -1$, and for $k \in \mathbb{N}$, we define

$$N_{2k-1} = \inf\{m > N_{2k-2} : X_m \le a\}, \qquad N_{2k} = \inf\{m > N_{2k-1} : X_m \ge b\}.$$

The above quantities $N_{2k-1}$,$N_{2k}$ are stopping times and the set containing values of $m$ in the transition from $a$ to $b$ can be defined as

$$H_m \triangleq \{N_{2k-1} < m \le N_{2k}\} = \{m - 1 \ge N_{2k-1}\} \cap \{m - 1 \ge N_{2k}\}^c \in \mathcal{F}_{m-1}.$$

Clearly, the event of $X$ being in an up crossing at time $m$ is predictable. The number of up crossings completed in time $n$ is

$$U_n = \sum_{m=1}^{n} H_m = \sup\{k : n \ge N_{2k}\}.$$

**Lemma 1.4 (Upcrossing inequality).** *If $X$ is a submartingale, then for $Y_n \triangleq a + (X_n - a)^+$, we have*

$$(b-a)\mathbb{E}U_n \leq \mathbb{E}Y_n - \mathbb{E}Y_0.$$

*Proof.* Since $X$ is a submartingale so is $Y$, as $Y_n$ is a convex function of $X_n$. Since each up crossing has a gain slightly more than $b-a$, the following inequality exists,

$$(b-a)U_n \leq (H\cdot Y)_n = \sum_{m=1}^{n} 1_{\{N_{2k-1}<m\leq N_{2k}\}}(Y_{m+1}-Y_m) = \sum_{k=1}^{U_n}(Y_{N_{2k+1}} - Y_{N_{2k+1}}).$$

Now let $K_m = 1 - H_m$, then $K$ is a predictable sequence, and

$$Y_n - Y_0 = (H\cdot Y)_n + (K\cdot Y)_n.$$

From the submartingale property of Y, it follows

$$\mathbb{E}[(K\cdot Y)_n] \geq \mathbb{E}[(K\cdot Y)_0] = 0.$$

Therefore, it follows that

$$\mathbb{E}(Y_n - Y_0) = \mathbb{E}(H\cdot Y)_n + \mathbb{E}(K\cdot Y)_n \geq \mathbb{E}(H\cdot Y)_n \geq (b-a)\mathbb{E}U_n.$$

□

**Theorem 1.5 (Martingale convergence theorem).** *If $\{X_n : n \in \mathbb{N}\}$ is a submartingale with $\sup_{n\in\mathbb{N}}\mathbb{E}X_n^+ < \infty$ then $\lim_{n\in\mathbb{N}} X_n = X$ a.s with $\mathbb{E}|X| < \infty$.*

*Proof.* Since $(X-a)^+ \leq X^+ + |a|$, it follows from upcrossing inequality that

$$\mathbb{E}U_n \leq \frac{\mathbb{E}X_n^+ + |a|}{b-a}.$$

The number of upcrossings $U_n$ increases with $n$, however the mean $\mathbb{E}U_n$ is bounded above for each $n \in \mathbb{N}$. Hence, $\lim_{n\in\mathbb{N}} \mathbb{E}U_n$ exists and is finite. Let $U := \lim_{n\in\mathbb{N}} U_n$ and since $\mathbb{E}U \leq \mathbb{E}[X_n^+] < \infty$, we have $U < \infty$ almost surely. This conclusion leads to

$$\Pr\{_{a,b\in\mathbb{Q}}\cup\{\liminf_{n\in\mathbb{N}} X_n < a < b < \limsup_{n\in\mathbb{N}} X_n\}\} = 0.$$

From the above probability, we have almost sure equality

$$\limsup_{n\in\mathbb{N}} X_n = \liminf_{n\in\mathbb{N}} X_n.$$

That is, the limit $\lim_{n\in\mathbb{N}} X_n$ exists almost surely. Fatou's lemma guarantees

$$\mathbb{E}X^+ \leq \liminf_{n\in\mathbb{N}} \mathbb{E}X_n^+ < \infty,$$

which implies $X < \infty$ almost surely. From the submartingale property of $X_n$, it follows that

$$\mathbb{E}X_n^- = \mathbb{E}X_n^+ - \mathbb{E}X_n \leq \mathbb{E}X_n^+ - \mathbb{E}X_0.$$

From Fatou's lemma, we get

$$\mathbb{E}X^- \leq \liminf_{n\in\mathbb{N}} \mathbb{E}X_n^- \leq \sup_{n\in\mathbb{N}} \mathbb{E}X_n^+ - \mathbb{E}X_0 < \infty.$$

This implies $X > -\infty$ almost surely, completing the proof. □