Lecture 27: Random Walks

1 Introduction

Let $X = \{X_n : n \in \mathbb{N}\}$ be a sequence of *iid* random variables with $\mathbb{E}|X_n| < \infty$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. Then the sequence $S = \{S_n : n \in \mathbb{N}_0\}$ is called a *random walk process*.

Example 1.1 (Simple random walk). Let $X_i \in \{-1, 1\}$ and $\Pr\{X_i = 1\} = p$, then random walk S is called a simple random walk.

Example 1.2. Stock prices each day can be modeled by a random walk.

Random walks are generalizations of renewal processes. If X was a sequence of non-negative random variables indicating inter-renewal times, then S_n is the instant of the nth renewal event.

2 Duality in random walks

Lemma 2.1 (Duality principle). *For any finite $n \in \mathbb{N}$, the joint distributions of finite sequence $(X_1, X_2, \cdots, X_n)$ and the reversed sequence $(X_n, X_{n-1}, \cdots, X_1)$ are identical.*

Proof. Since X is a sequence of *iid* random variables, it is exchangeable. The reversed sequence is $(X_{\pi(1)}, \dots, X_{\pi(n)})$ where $\pi : [n] \to [n]$ is permutation with $\pi(i) = n - i + 1$. □

Corollary 2.2. *Distribution of S_k and $S_n - S_{n-k}$ are identical for any $k \in [n]$.*

Proof. Using duality principle, we can write the following equality for any $x \in \mathbb{R}, k \in [n]$

$$\Pr\{S_k \le x\} = \Pr\{\sum_{i=1}^k X_i \le x\} = \Pr\{\sum_{i=1}^k X_{n-i+1} \le x\} = \Pr\{\sum_{i=n-k+1}^n X_i \le x\} = \Pr\{S_n - S_{n-k} \le x\}.$$

□

Proposition 2.3. *Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of iid random variables with positive mean. Let $S_n = \sum_{k=1}^n X_i$ be a random walk with step size X_n. If $N = \min\{n \in \mathbb{N} : S_n > 0\}$, then $\mathbb{E}N < \infty$.*

Proof. Consider a discrete process $\{T_k \in \mathbb{N}_0 : k \in \mathbb{N}_0\}$, where $T_0 = 0$ and for each $k \in \mathbb{N}_0$

$$T_{k+1} = \inf\{n > T_k : S_n \le S_{N_k}\} = N_k + \inf\{n \in \mathbb{N} : S_{N_k+n} \le S_{N_k}\}.$$

Since for any finite $n \in \mathbb{N}$, the distribution of $(X_1, \dots, X_n)$ is identical to that of $(X_{T_k+1}, \dots, X_{T_k+n})$, and

$$T_{k+1} - T_k = \inf\{n \in \mathbb{N} : \sum_{i=1}^n X_{T_k+i} \le 0\},$$

we observe that $T_k - T_{k-1}$ are *iid* for each $k \in \mathbb{N}$, with complementary distribution

$$\bar{F}(m) = \Pr\{T_{k+1} - T_k > m\} = \Pr\{S_1 > 0, S_2 > 0, \dots, S_m > 0\}.$$

Therefore, $\{T_k : k \in \mathbb{N}_0\}$ is a renewal process and at each renewal instant $\{T_k = n\}$,

$$S_n \leq S_{n-1}, S_n \leq S_{n-2}, \dots, S_n \leq 0.$$

Hence, T_k denotes the kth renewal instant corresponding to the random walk S_n hitting kth low. We can define the inverse counting process $\{N_n \in \mathbb{N}_0 : n \in \mathbb{N}_0\}$ for this renewal process as

$$N_n = \sum_{j=1}^{n} 1_{\{T_j \leq n\}}, \qquad \text{or } \{N_n \geq k\} = \{T_k \leq n\}.$$

From definition of stopping time N and duality principle, we can write

$$\Pr\{N > n\} = \Pr\{S_1 \leq 0, \dots, S_n \leq 0\} = \Pr\{S_n \leq S_{n-1}, \dots, S_n \leq 0\}.$$

The event of renewal process hitting a new low at n is same as some renewal occurring at time n. That is,

$$\sum_{n \in \mathbb{N}} 1_{\{S_n \leq S_{n-1}, S_n \leq S_{n-2}, \dots, S_n \leq S_0\}} = \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} 1_{\{T_k = n\}} = \sum_{k \in \mathbb{N}} \sum_{n \geq k} 1_{\{T_k = n\}} = \sum_{k \in \mathbb{N}} 1_{\{T_k \leq \infty\}} = N_\infty.$$

Therefore, we can write the mean of stopping time N as

$$\mathbb{E}N = 1 + \sum_{n \in \mathbb{N}} \Pr\{N > n\} = 1 + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} \Pr\{T_k = n\} = 1 + \sum_{k \in \mathbb{N}} \sum_{n \geq k} \Pr\{T_k = n\} = 1 + \mathbb{E}N_\infty.$$

Since $\mathbb{E}X_1 > 0$, it follows from strong law of large numbers that $S_n \to \infty$. Hence, the expected number of renewals that occur is finite. Thus $\mathbb{E}N < \infty$. □

The number of distinct values of $(S_0, \cdots, S_n)$ is called **range**, denoted by R_n We define by stopping time T_k, first return of random walk to k

$$T_k = \inf\{n \in \mathbb{N} : S_n = k\}.$$

Proposition 2.4. *For a simple random walk,*

$$\lim_{n \in \mathbb{N}} \frac{\mathbb{E}R_n}{n} = \Pr\{T_0 > \infty\}.$$

Proof. We can define indicator function for S_k being a distinct number from $S_1, \dots, S_{k-1}$, as

$$I_k = 1_{\{S_k \neq S_{k-1}, \dots, S_k \neq S_1\}}.$$

Then, we can write range R_n in terms of indicator I_k as

$$R_n = 1 + \sum_{k=1}^{n} I_k$$

Further, using the duality principle, we can write

$$\mathbb{E}R_n = 1 + \sum_{k=1}^{n} \Pr\{S_1 \neq 0, \dots, S_k \neq 0\} = \sum_{k=0}^{n} \Pr\{T_0 > k\}$$

Result follows by dividing both sides by n and taking limits. □

2.1 Simple random walk

Theorem 2.5 (range). *For a simple random walk with* $\Pr\{X_1 = 1\} = p$*, the following holds*

$$\lim_{n\in\mathbb{N}} \frac{\mathbb{E}R_n}{n} = \begin{cases} 2p-1, & p > \frac{1}{2} \\ 2(1-p)-1, & p \leq \frac{1}{2}. \end{cases}$$

Proof. When $p = \frac{1}{2}$, this random walk is recurrent and thus

$$\Pr\{T_0 > \infty\} = 0 = \lim_{n\in\mathbb{N}} \frac{\mathbb{E}R_n}{n}.$$

For$p > \frac{1}{2}$, let $\alpha = \Pr\{T_0 < \infty | X_1 = 1\}$. Since $\mathbb{E}X > 0$, we know that $S_n \to \infty$ and hence $\Pr\{T_0 < \infty | X_1 = -1\} = 1$. We can write unconditioned probability of return of random walk to 0 as

$$\Pr\{T_0 < \infty\} = \alpha p + 1 - p.$$

Conditioning on X_2 and from strong law of large numbers, we yield

$$\alpha = \Pr\{T_0 < \infty | X_1 = 1\} = p \Pr\{T_0 < \infty | S_2 = 2\} + (1-p).$$

From Markov property and homogeneity of random walk process, it follows that

$$\Pr\{T_0 < \infty | S_2 = 2\} = \Pr\{T_0 < \infty | S_{T_1} = 1, T_1 < \infty\} \Pr\{T_1 < \infty | S_2 = 2\} = \alpha^2.$$

We conclude $\alpha = \alpha^2 p + 1 - p$, and since $\alpha < 1$ due to transience, we get $\alpha = \frac{1-p}{p}$, and hence the result follows. We can show similarly for the case when $p < 1/2$. □