Lecture 28: Random walks

1 GI/GI/1 Queueing Model

Consider a $GI/GI/1$ queue. Customers arrive in accordance with a renewal process having an arbitrary interarrival distribution F, and the service distribution G.

Proposition 1.1. *Let D_n be the delay in the queue of the n^{th} customer in a GI/GI/1 queue with independent inter-arrival times X_n and service times Y_n. Let S_n be a random walk with iid steps $U_n = Y_n - X_{n+1}$ for all $n \in \mathbb{N}$. Then, we can write*

$$\Pr\{D_{n+1} \geq c\} = \Pr\{S_j \geq c, \textit{ for some } j \in [n]\}. \tag{1}$$

Proof. The following recursion for D_n is easy to verify

$$D_{n+1} = (D_n + Y_n - X_{n+1})1_{\{D_n+Y_n-X_{n+1}\geq 0\}} = \max\{0, D_n + U_n\}.$$

Iterating the above relation with $D_1 = 0$ yields

$$D_{n+1} = \max\{0, U_n + \max\{0, D_{n-1} + U_{n-1}\}\} = \max\{0, U_n, U_n + U_{n-1} + D_{n-1}\}.$$

For the random walk S_n with steps U_n, we can write delay in terms of random walk S_n as

$$D_{n+1} = \max\{0, S_n - S_{n-1}, S_n - S_{n-2}, \dots, S_n - S_0\}.$$

Using the duality principle, we can rewrite the following equality for delay in distribution

$$D_{n+1} = \max\{0, S_1, S_2, \dots, S_n\}.$$

□

Corollary 1.2. *If $\mathbb{E}U_n \geq 0$, then for all c, we have* $\Pr\{D_\infty \geq c\} \triangleq \lim_{n\in\mathbb{N}} \Pr\{D_n \geq c\} = 1$.

Proof. It follows from Proposition 1.1 that $\Pr\{D_{n+1} \geq c\}$ is nondecreasing in n. Hence, by MCT the limit exists and is denoted by $\Pr\{D_\infty \geq c\} = \lim_{n\in\mathbb{N}} \Pr\{D_n \geq c\}$. Therefore, by continuity of probability, we have from (1), that

$$\Pr\{D_\infty \geq c\} = \Pr\{S_n \geq c \text{ for some } n\}. \tag{2}$$

If $\mathbb{E}U_n = \mathbb{E}Y_n - \mathbb{E}X_{n+1}$ is positive, then by strong law of large numbers the random walk S_n will converge to positive infinity with probability 1. The above will also be true when $E[U_n] = 0$, then the random walk is recurrent. □

Remark 1.3. Hence, we get that $\mathbb{E}Y_n < \mathbb{E}X_{n+1}$ implies the existence of a stationary distribution.

Proposition 1.4 (Spitzer's Identity). *Let $M_n = \max\{0, S_1, S_2, \dots, S_n\}$ for $n \in \mathbb{N}$, then*

$$\mathbb{E}M_n = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}S_k^+.$$

Proof. We can decompose M_n as

$$M_n = 1_{\{S_n > 0\}} M_n + 1_{\{S_n \le 0\}} M_n.$$

We can rewrite first term in decomposition as,

$$1_{\{S_n>0\}} M_n = 1_{\{S_n>0\}} \max_{i \in [n]} S_i = 1_{\{S_n>0\}} (X_1 + \max\{0, S_2 - S_1, \dots, S_n - S_1\})$$

Hence, taking expectation and using exchangeability, we get

$$\mathbb{E}[M_n 1_{\{S_n>0\}}] = \mathbb{E}[X_1 1_{\{S_n>0\}}] + \mathbb{E}[M_{n-1} 1_{\{S_n>0\}}].$$

Since X_i, S_n has the same joint distribution for all i,

$$\mathbb{E}S_n^+ = \mathbb{E}[S_n 1_{\{S_n>0\}}] = \mathbb{E} \sum_{i=1}^{n} X_i 1_{\{S_n>0\}}] = n\mathbb{E}[X_1 1_{\{S_n>0\}}].$$

Therefore, it follows that

$$\mathbb{E}[1_{\{S_n>0\}} M_n] = \mathbb{E}[1_{\{S_n>0\}} M_{n-1}] + \frac{1}{n} \mathbb{E}[S_n^+].$$

Also, $S_n \le 0$ implies that $M_n = M_{n-1}$, it follows that

$$1_{\{S_n \le 0\}} M_n = 1_{\{S_n \le 0\}} M_{n-1}.$$

Thus, we obtain the following recursion,

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n-1}] + \frac{1}{n} \mathbb{E}[S_n^+].$$

Result follow from the fact that $M_1 = S_1^+$. □

Remark 1.5. Since $D_{n+1} = M_n$, we have $\mathbb{E}[D_{n+1}] = \mathbb{E}[M_n] = \sum_{k=1}^{n} \frac{1}{k} E[S_k^+]$.

2 Martingales for Random Walks

Proposition 2.1. *A random walk S_n with step size $X_n \in [-M, M] \cap \mathbb{Z}$ for some finite M is a recurrent DTMC iff $\mathbb{E}X = 0$.*

Proof. If $\mathbb{E}X \neq 0$, the random walk is clearly transient since, it will diverge to $\pm\infty$ depending on the sign of $\mathbb{E}X$. Conversely, if $\mathbb{E}X = 0$, then S_n is a martingale. Assume that the process starts in state i. We define

$$A = \{-M, -M+1, \cdots, -2, -1\}, \qquad A_j = \{j+1, \dots, j+M\}, \ j > i.$$

Let N denote the hitting time to A or A_j by random walk S_n. Since N is a stopping time and $S_{N \wedge n} \le |M| + j$, by optional stopping theorem, we have

$$\mathbb{E}_i[S_N] = \mathbb{E}_i[S_0] = i.$$

Thus we have

$$i = \mathbb{E}_i[S_N] \geq -M\mathbb{P}_i\{S_N \in A\} + j(1 - \mathbb{P}_i\{S_N \in A\}).$$

Rearranging this, we get a bound on probability of random walk S_n hitting A over A_j as

$$\mathbb{P}_i\{S_n \in A \text{ for some } n\} \geq \mathbb{P}_i\{S_N \in A\} \geq \frac{j-i}{j+M}.$$

Taking limit $j \to \infty$, we see that for any $i \geq 0$, we have $\mathbb{P}_i\{S_n \in A \text{ for some } n\} = 1$. Similarly, taking $B = \{1, 2, \cdots, M\}$, we can show that for any $i \geq 0$, $\mathbb{P}_i\{S_n \in B \text{ for some } n\} = 1$. Result follows from combining the above two arguments to see that for any $i \geq 0$,

$$\mathbb{P}_i\{S_n \in A \cup B \text{ for some } n\} = 1.$$

□

Proposition 2.2. *Consider a random walk S_n with mean step size $\mathbb{E}[X] \neq 0$. For $A, B > 0$, let P_A denote the probability that the walk hits a value greater than A before it hits a value less than $-B$. Then, for $\theta \neq 0$ such that $\mathbb{E}e^{\theta X_1} = 1$, we have*

$$P_A \approx \frac{1 - e^{-\theta B}}{e^{\theta A} - e^{-\theta B}}.$$

Approximation is an equality when step size is unity and A and B are integer valued.

Proof. For any $A, B > 0$, we can define stopping times

$$T_A = \inf\{n \in \mathbb{N} : S_n \geq a\}, \qquad T_{-B} = \inf\{n \in \mathbb{N} : S_n \leq -B\}.$$

We are interested in computing the probability

$$P_A = \Pr\{T_A < T_{-B}\}.$$

Now let $Z_n = e^{\theta S_n}$. We can see that Z_n is a martingale with mean 1. Define a stopping time $N = T_A \wedge T_{-B}$. From Doob's Theorem, $\mathbb{E}[e^{S_N}] = 1$. Thus we get

$$1 = \mathbb{E}[e^{\theta S_N} | S_N \geq A] P_A + \mathbb{E}[e^{\theta S_N} | S_N \leq -B](1 - P_A).$$

We can obtain an approximation for P_A by neglecting the overshoots past A or $-B$. Thus we get

$$\mathbb{E}[e^{\theta S_N} | S_N \geq A] \approx e^{\theta A}, \qquad E[e^{\theta S_N} | S_N \leq -B] \approx e^{-\theta B}.$$

The result follows. □