Lecture-01: Sample and Event Space

1 Random Experiment

Consider an experiment where the outcomes are random and unpredictable.

Definition 1.1 (Sample space). The set of all possible outcomes of a random experiment is called **sample space** and denoted by Ω.

Let $[N] \triangleq \{1,2,\dots,N\}$. We denote the cardinality of a set A by $|A|$. Any set which is bijective to the set $[N]$ has cardinality N. Any set which has a finite cardinality is called a countably finite set. We denote the set of natural numbers by $\mathbb{N} \triangleq \{1,2,\dots\}$. Any set which is bijective to the set of natural numbers $\mathbb{N}$ is called a countably infinite set.

Definition 1.2 (Event space). A collection of subsets of sample space Ω is called the **event space** if it is a σ-algebra over subsets of Ω, and denoted by $\mathcal{F}$. In other words, the collection $\mathcal{F}$ satisfies the following properties. are satisfied.

1. Event space includes the certain event Ω. That is, $\Omega \in \mathcal{F}$.
2. Event space is closed under complements. That is, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. Event space is closed under countable unions. That is, if $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $\cup_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

The elements of the event space $\mathcal{F}$ are called **events**.

Remark 1. Not all subsets of the sample space Ω belong to the event space.

Remark 2. From the inclusion of certain event and the closure complements of σ-algebras, it follows that the impossible event $\emptyset \in \mathcal{F}$. That is, since $\Omega \in \mathcal{F}$ and $\emptyset = \Omega^c$, we have $\emptyset \in \mathcal{F}$.

Remark 3. Event space is closed under finite unions. In particular, if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$. This follows from the fact that we can $A_1 = A, A_2 = B$ and $A_i = \emptyset$ for all $i \geqslant 2$, and apply closure under countable unions.

Remark 4. Event space is closed under countable intersections. That is, $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $\cap_{i\in\mathbb{N}} A_i \in \mathcal{F}$. To show this, we first notice that $A_i^c \in \mathcal{F}$ for all $i \in \mathbb{N}$ from the closure under complements. Further, we have $\cup_{i\in\mathbb{N}} A_i^c \in \mathcal{F}$ from the closure under countable unions, and then the result follows from taking the complement and the closure under complements.

Example 1.3 (Coin toss). Consider a single coin toss, where the outputs can be heads or tails, denoted by H and T respectively. In this case, the sample space is $\Omega = \{H, T\}$ and the event space is $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$. Can you verify that $\mathcal{F}$ is a σ-algebra?

For sets A, B, the collection A^B is the collection of all A-valued functions with the domain B. That is, any element $f \in A^B$ is a function $f : B \to A$ and $f(b) \in A$ for all $b \in B$.

Exercise 1.4. Show the following are true.

1. $|A^B| = |A|^{|B|}$.
2. $A^{[N]}$ is set of all A-valued N-length sequences.
3. $A^{\mathbb{N}}$ is a set of all A-valued countably infinite sequences indexed by the set of natural numbers $\mathbb{N}$.

Example 1.5 (Finite coin tosses). Consider N tosses of a single coin, where the possible output of each coin toss belongs to the set $\{H,T\}$ as before. In this case, the sample space is

$$\Omega = \{H,T\}^{[N]} = \{(\omega_1,\dots,\omega_N) : \omega_i \in \{H,T\} \text{ for all } i \in [N]\}.$$

The event space is $\mathcal{F} = 2^{\Omega} = \{A : A \subseteq \Omega\}$. Can you verify that $\mathcal{F}$ is a σ-algebra?

Example 1.6 (Countably infinite coin tosses). A single coin is tossed countably infinite times. A single outcome of this repeated coin toss can be denoted by a sequence $\omega = (\omega_1,\omega_2,\dots)$, where $\omega_i \in \{H,T\}$ is the output of ith coin toss. Then, we can write the sample space as

$$\Omega = \{H,T\}^{[N]} = \{(\omega_1,\omega_2,\dots) : \omega_i \in \{H,T\} \text{ for all } i \in \mathbb{N}\}.$$

We will construct an event space $\mathcal{F}$ on this outcome space, which would not be the power set of the sample space. By definition, we must have the certain and the impossible event in an event space. We consider the event space $\mathcal{F}$ generated by the events

$$A_n \triangleq \{\omega \in \Omega : \omega_i = H \text{ for some } i \in [n]\}, \text{ for each } n \in \mathbb{N}.$$

From the closure under countable additivity, we have $\cup_{n\in\mathbb{N}} A_n = \Omega \in \mathcal{F}$. We see that $(A_n \in \mathcal{F} : n \in \mathbb{N})$ is a sequence of increasing events.

Let B_n be the event of observing first head in nth toss, then we will show that $B_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. We see that the event $B_1 = A_1$ and the event B_{n+1} can be written as

$$B_{n+1} = \{\omega \in \Omega : \omega_i = T \text{ for } i \leqslant n, \omega_{n+1} = H\} = A_{n+1} \setminus A_n.$$

Therefore, $B_n \in \mathcal{F}$ for all $n \in \mathbb{N}$.