# Lecture-02: Probability Function

## 1 Probability Function

Consider $N$ trials of a random experiment over outcome space $\Omega$ and the event space $\mathcal{F}$. Let $X_n \in \Omega$ denote the outcome of the experiment of the $n$th trial. We define the indicator function

$$\mathbb{1}_{\{X_n \in A\}} = \begin{cases} 1, & X_n \in A, \\ 0, & X_n \notin A. \end{cases}$$

Let $N(A)$ denote the number of times an event $A$ occurs in $N$ trials, then $N(A) = \sum_{n=1}^{N} \mathbb{1}_{\{X_n \in A\}}$. We denote the relative frequency of an event $A$ in $N$ trials by $\frac{N(A)}{N}$. We observe the following properties of the relative frequency.

1. For all $A \in \mathcal{F}$, we have $0 \leqslant \frac{N(A)}{N} \leqslant 1$. This follows from the fact that $0 \leqslant N(A) \leqslant N$ for any $A \in \mathcal{F}$.
2. Suppose $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then $\frac{N(\cup_{i\in\mathbb{N}} A_i)}{N} = \sum_{i\in\mathbb{N}} \frac{N(A_i)}{N}$. This follows from the fact that for disjoint events $(A_i : i \in \mathbb{N})$, we have
$$\mathbb{1}_{\{X_n \in \cup_{i\in\mathbb{N}} A_i\}} = \sum_{i\in\mathbb{N}} \mathbb{1}_{\{X_n \in A_i\}}.$$
3. For the certain event $\Omega$, we have $\frac{N(\Omega)}{N} = 1$. This follows from the fact that $N(\Omega) = N$.

Since the relative frequency is positive bounded, it may converge to a real number as $N$ grows very large, and the limit $\lim_{N\to\infty} \frac{N(A)}{N}$ may exist. Inspired by the relative frequency, we list the following axioms for a probability function $P : \mathcal{F} \to [0,1]$.

**Axiom 1.1 (Axioms of probability).** We define a probability measure on sample space $\Omega$ and event space $\mathcal{F}$ by a function $P : \mathcal{F} \to [0,1]$ which satisfies the following axioms.

**Non-negativity** For all events $A \in \mathcal{F}$, we have $P(A) \geqslant 0$.

**$\sigma$-additivity** For an infinite sequence of mutually disjoint events $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, we have $P(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} P(A_i)$.

**Certainity** $P(\Omega) = 1$.

**Definition 1.2 (Limits of monotonic sets).** For a sequence of non-decreasing sets $(A_n : n \in \mathbb{N})$, we can define the limit as

$$\lim_{n\to\infty} A_n \triangleq \cup_{n\in\mathbb{N}} A_n.$$

Similarly, for a sequence of non-increasing sets $(A_n : n \in \mathbb{N})$, we can define the limit as

$$\lim_{n\to\infty} A_n \triangleq \cap_{n\in\mathbb{N}} A_n.$$

**Example 1.3 (Monotone sets).** Consider $(-\frac{1}{n} \in \mathbb{R} : n \in \mathbb{N})$ a monotonically increasing sequence that converges to the limit 0. We consider sequence of sets $(A_n = [-2, -\frac{1}{n}] : n \in \mathbb{N})$ and $(B_n = [-2, \frac{1}{n}] : n \in \mathbb{N})$, which are monotonically increasing and decreasing respectively. We can verify the following limits

$$\lim_n A_n = \cup_{n\in\mathbb{N}} A_n = [-2, 0), \qquad \lim_n B_n = \cap_{n\in\mathbb{N}} B_n = [-2, 0].$$

**Definition 1.4 (Limits of sets).** For a sequence of sets $(A_n : n \in \mathbb{N})$, we can define the limit superior and limit inferior of this sequence of sets as

$$\limsup_{n\to\infty} A_n \triangleq \cap_{n\in\mathbb{N}} \cup_{m\geqslant n} A_m = \lim_{n\to\infty} \cup_{m\geqslant n} A_m, \qquad \liminf_{n\to\infty} A_n \triangleq \cup_{n\in\mathbb{N}} \cap_{m\geqslant n} A_m = \lim_{n\to\infty} \cap_{m\geqslant n} A_m.$$

In general, $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$. If the limit superior and limit inferior of any sequence of sets are equal, then the limit is defined as

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

**Example 1.5 (Sequence of sets with different limits).** We consider sequence of sets $(A_n = [-2, (-1)^n + \frac{1}{n}] : n \in \mathbb{N})$. It follows that $F_n = [-2, -1]$ and

$$E_n = \cup_{m\geqslant n} = \begin{cases} [-2, 1 + \frac{1}{n+1}], & n \text{ odd}, \\ [-2, 1 + \frac{1}{n}], & n \text{ even}. \end{cases}$$

We can verify the following limits

$$\liminf_n A_n = \cup_{n\in\mathbb{N}} F_n = [-2, -1], \qquad \limsup_n A_n = \cap_{n\in\mathbb{N}} F_n = [-2, 1].$$

**Lemma 1.6.** *For a sequence of sets $(A_n : n \in \mathbb{N})$, we have $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.*

*Proof.* For each $n \in \mathbb{N}$, we define $E_n \triangleq \cup_{m\geqslant n} A_m$ and $F_n \triangleq \cap_{m\geqslant n} A_m$. We see that $F_{n_0} \subseteq A_m$ for all $m \geqslant n_0$. In particular, we can write $\cup_{n\in\mathbb{N}} F_n \subseteq \cup_{m\geqslant n} A_m$ for each $n \in \mathbb{N}$, and hence the result follows. □

**Theorem 1.7.** *For any probability space $(\Omega, \mathcal{F}, P)$, we have the following properties of probability measure.*

***impossibility:*** *$P(\emptyset) = 0$.*

***finite additivity:*** *For disjoint events $(A_1, \dots, A_n) \in \mathcal{F}^n$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, we have $P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$.*

***monotonicity:*** *If events $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $P(A) \leqslant P(B)$.*

***inclusion-exclusion:*** *For any events $A, B \in \mathcal{F}$, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.*

***continuity:*** *For an increasing sequence of events $(A_i \in \mathcal{F} : i \in \mathbb{N})$ such that $\lim_n A_n$ exists, we have $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$.*

*Proof.* We consider the probability space $(\Omega, \mathcal{F}, P)$.

1. We take disjoint events $(E_i : i \in \mathbb{N})$ where $E_1 = \Omega$ and $E_i = \emptyset$ for $i \geqslant 2$. It follows that $\cup_{i\in\mathbb{N}} E_i = \Omega$ and $(E_i : i \in \mathbb{N})$ is a collection of mutually disjoint events. From the countable additivity axiom of probability, it follows that

$$P(\Omega) = P(\Omega) + \sum_{i\geqslant 2} P(E_i).$$

   Since $P(E_i) \geqslant 0$, it implies that $P(\emptyset) = 0$.

2. We see that finite additivity follows from the countable additivity. We consider disjoint events $A_1, \dots, A_n$, and take $A_i = \emptyset$ for all $i > n$. It follows that the sequence of sets $(A_i : i \in \mathbb{N})$ is mutually disjoint, and since $P(\emptyset) = 0$, it follows that

$$P(\sum_{i\in\mathbb{N}} A_i) = P(\sum_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) + \sum_{i>n} P(\emptyset) = \sum_{i=1}^n P(A_i).$$

3. For events $A, B \in \mathcal{F}$ such that $A \subseteq B$, we can take disjoint events $E_1 = A$ and $E_2 = B \setminus A$. From closure under complements and intersection, it follows that $E_2 \in \mathcal{F}$. From non-negativity of probability, we have $P(E_2) \geqslant 0$. Finally, the result follows from finite additivity of disjoint events

$$P(B) = P(E_1 \cup E_2) = P(E_1) + P(E_2) \geqslant P(A).$$

4. For any two events $A, B \in \mathcal{F}$, we can write the following events as disjoint unions

$$A = (A \setminus B) \cup (A \cap B), \qquad B = (B \setminus A) \cup (A \cap B), \qquad A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).$$

The result follows from the finite additivity of probability of disjoint events.

5. We show the continuity for non-decreasing and non-increasing sequence of sets.
**Continuity for increasing sets.** Let $(A_n \in \mathcal{F} : n \in \mathbb{N})$ be a non-decreasing sequence of events, then $\lim_{n\to\infty} A_n = \cup_{n \in \mathbb{N}} A_n$. This implies that $(P(A_n) : n \in \mathbb{N})$ is a non-negative non-decreasing bounded sequence, and hence as a limit. It remains to show that $\lim_{n\to\infty} P(A_n) = P(\cup_{n\in\mathbb{N}} A_n)$. To this end, we observe that $(A_1, A_2 \setminus A_1, \dots, A_n \setminus A_{n-1}, \dots)$ is a sequence of disjoint sets, with union $A_n$ and $P(A_i \setminus A_{i-1}) = P(A_i) - P(A_{i-1})$. Therefore, we can write for each $n \in \mathbb{N}$

$$P(A_n) = P(A_1) + \sum_{i=1}^{n-1} P(A_i \setminus A_{i-1}) = P(A_1) + \sum_{i=1}^{n-1} (P(A_i) - P(A_{i-1})).$$

Hence, we can write for $\lim_{n\to\infty} A_n = \cup_{n\in\mathbb{N}} A_n$,

$$P(\cup_{n\in\mathbb{N}} A_n) = P(A_1) + \sum_{i\in\mathbb{N}} (P(A_i) - P(A_{i-1})) = P(A_1) + \lim_{n\to\infty} \sum_{i=1}^{n} (P(A_i) - P(A_{i-1})) = \lim_{n\to\infty} P(A_n).$$

**Continuity for decreasing sets.** Similarly, for a non-increasing sequence of sets $(B_n \in \mathcal{F} : n \in \mathbb{N})$, we can find the non-decreasing sequence of sets $(B_n^c \in \mathcal{F} : n \in \mathbb{N})$. By the first part, we have

$$P(\lim_{n\to\infty} B_n) = P(\cap_{n\in\mathbb{N}} B_n) = 1 - P(\cup_{n\in\mathbb{N}} B_n^c) = 1 - P(\lim_{n\to\infty} B_n^c) = 1 - \lim_{n\to\infty} P(B_n^c) = \lim_{n\to\infty} P(B_n).$$

**Continuity for general sequence of sets.** We can similarly prove the general result for a sequence of sets $(A_n \in \mathcal{F} : n \in \mathbb{N})$ such that the limits $\lim_n A_n$ exists. We can define non-increasing sequences of sets $(E_n = \cup_{m \geqslant n} A_m \in \mathcal{F} : n \in \mathbb{N})$ and non-decreasing sequences of sets $(F_n = \cap_{m\geqslant n} A_m \in \mathcal{F} : n \in \mathbb{N})$. From the continuity of probability for the monotonic sets, we have

$$P(\limsup_n A_n) = P(\cap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} P(E_n), \qquad P(\liminf_n A_n) = P(\cup_{n\in\mathbb{N}} F_n) = \lim_{n\to\infty} P(F_n).$$

From the definition of two sequences of sets, we obtain

$$P(E_n) \geqslant \sup_{m\geqslant n} P(A_m), \qquad P(F_n) \leqslant \inf_{m\geqslant n} P(A_m).$$

Therefore taking limsup and liminf, we obtain

$$\limsup_{n\in\mathbb{N}} P(E_n) \geqslant \inf_{n\in\mathbb{N}} \sup_{m\geqslant n} P(A_m) \geqslant \sup_{n\in\mathbb{N}} \inf_{m\geqslant n} P(A_m) \geqslant \liminf_{n\in\mathbb{N}} P(F_n).$$

Since $P(\lim_n A_n) = \lim_n P(E_n) = \lim_n P(F_n)$ exists, the result follows.

□