Lecture-04: Random Variable

1 Random Variable

Definition 1.1 (Random variable). Consider a probability space $(\Omega, \mathcal{F}, P)$. A **random variable** $X : \Omega \to \mathbb{R}$ is a real-valued function from the sample space to real numbers, such that for each $x \in \mathbb{R}$ the event

$$A(x) \triangleq \{\omega \in \Omega : X(\omega) \leqslant x\} = X^{-1}(-\infty, x] \in \mathcal{F}.$$

We say that the random variable X is $\mathcal{F}$-measurable and the probability of this event is denoted by

$$F_X(x) \triangleq P(A(x)) = P(\{X \leqslant x\}) = P \circ X^{-1}(-\infty, x].$$

The function $F_X : \mathbb{R} \to [0,1]$ is called the **distribution function** (CDF) of a random variable X.

Remark 1. Since any outcome $\omega \in \Omega$ is random, so is the real value $X(\omega)$.

Remark 2. Probability is defined only for events and not for random variables. The events of interest for random variables are the upper-level sets $A(x)$ for any real x.

Definition 1.2 (Borel sets). The smallest σ-algebra generated by the half-open sets $(-\infty, x]$ for all $x \in \mathbb{R}$ is called a **Borel σ-algebra** and denoted by $\mathcal{B}(\mathbb{R})$. The elements of the Borel σ-algebra are called **Borel sets**.

Remark 3. The event space generated by a random variable is collection of the inverse of Borel sets.

Lemma 1.3 (Properties of distribution function). *The distribution function F_X for any random variable X satisfies the following properties.*

1. *The distribution function is monotonically non-decreasing in $x \in \mathbb{R}$.*
2. *The distribution function is right-continuous at all points $x \in \mathbb{R}$.*
3. *The upper limit is $\lim_{x \to \infty} F_X(x) = 1$ and the lower limit is $\lim_{x \to -\infty} F_X(x) = 0$.*

Proof. Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$.

1. Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leqslant x_2$. Then $A(x_1) \subseteq A(x_2)$ and from the monotonicity of the probability the result follows.
2. For any $x \in \mathbb{R}$, consider any monotonically decreasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = x$. It follows that the sequence of events $\left(A(x_n) = X^{-1}(-\infty, x_n] \in \mathcal{F} : n \in \mathbb{N}\right)$, is monotonically decreasing and hence $\lim_{n \in \mathbb{N}} A(x_n) = \cap_{n \in \mathbb{N}} A(x_n) = A(x)$. The right-continuity then follows from the continuity of probability, since
$$F_X(x) = P(A(x)) = P(\lim_{n \in \mathbb{N}} A(x_n)) = \lim_{n \in \mathbb{N}} P(A(x_n)) = \lim_{x_n \downarrow x} F(x_n).$$
3. Consider a monotonically increasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = \infty$, then $(A(x_n) \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically increasing sequence of sets and $\lim_n A(x_n) = \cup_{n \in \mathbb{N}} A(x_n) = \Omega$. From the continuity of probability, it follows that
$$\lim_{x_n \to \infty} F_X(x_n) = \lim_n P(A(x_n)) = P(\lim_n A(x_n)) = P(\Omega) = 1.$$
Similarly, we can take a monotonically decreasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = -\infty$, then $(A(x_n) \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically decreasing sequence of sets and $\lim_n A(x_n) = \cap_{n \in \mathbb{N}} A(x_n) = \emptyset$. From the continuity of probability, it follows that $\lim_{x_n \to -\infty} F_X(x_n) = 0$.

□

Remark 4. If two reals $x_1 < x_2$ then $F_X(x_1) \leqslant F_X(x_2)$ with equality if and only if $P\{(x_1 < X \leqslant x_2\}) = 0$.

Example 1.4 (Constant random variable). Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $X(\omega) = c$ for all outcomes $\omega \in \Omega$. The distribution function is a right-continuous step function at c with step-value unity. That is,

$$F_X(x) = \mathbb{1}_{\{x \geqslant c\}}.$$

We observe that $P(X = c) = 1$. Does this make sense? Is $\{\omega \in \Omega : X(\omega) = x\}$ an event for a general random variable?

Consider a monotonically increasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = x$. Then, the sequence of events $(A(x_n) \in \mathcal{F} : n \in \mathbb{N})$ is monotonically increasing and hence the $\lim_n A(x_n) = \{X < x\} \in \mathcal{F}$ is an event. It follows that $\{X = x\} = A(x) \cap \{X < x\}^c \in \mathcal{F}$ is also an event.

Definition 1.5. For a continuous random variable X, there exists **density function** $f_X : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u) du.$$

Example 1.6 (Gaussian random variable). For a probability space $(\Omega, \mathcal{F}, P)$, **Gaussian random variable** is a continuous random variable $X : \Omega \to \mathbb{R}$ defined by its density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ x \in \mathbb{R}.$$

1.1 Discrete random variables

Definition 1.7 (Discrete random variables). If a random variable $X : \Omega \to \mathcal{X} \subseteq \mathbb{R}$ takes countable values on real-line, then it is called a **discrete random variable**. That is, the range of random variable $\mathcal{X}$ is countable, and the random variable is completely specified by the **probability mass function**

$$P_X(x) = P(\{X = x\}), \text{ for all } x \in \mathcal{X}.$$

Definition 1.8 (Indicator functions). For a probability space $(\Omega, \mathcal{F}, P)$ and any event $A \in \mathcal{F}$, we can define an **indicator function** $\mathbb{1}_A : \Omega \to \{0, 1\}$ as

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Remark 5. Indicator function is a discrete random variable.

Definition 1.9 (Simple functions). For a probability space $(\Omega, \mathcal{F}, P)$, a finite n, events $A_1, \dots, A_n \in \mathcal{F}$, and real constants $x_1, \dots, x_n$ we can define a simple function $X : \Omega \to \{x_1, \dots, x_n\}$ to be

$$X(\omega) = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}(\omega).$$

Lemma 1.10. *Any discrete random variable is a linear combination of indicator function over a partition of the sample space.*

Proof. For a discrete random variable $X : \Omega \to \mathcal{X} \subset \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$, the range $\mathcal{X}$ is countable, and we can define sets $A_x \triangleq \{\omega \in \Omega : X(\omega) = x\}$ for each $x \in \mathcal{X}$. Then $(A_x \subseteq \Omega : x \in \mathcal{X})$ partition the sample space Ω, and are events from the definition of random variables. We can write

$$X(\omega) = \sum_{x \in \mathcal{X}} x \mathbb{1}_{A_x}(\omega).$$

□

Example 1.11 (Bernoulli random variable). For the probability space $(\Omega, \mathcal{F}, P)$, the **Bernoulli random variable** is a mapping $X : \Omega \to \{0, 1\}$ and $P_X(1) = p$. The distribution function F_X is given by

$$F_X = (1 - p)\mathbb{1}_{[0,1)} + \mathbb{1}_{[1,\infty)}.$$