Lecture-11: Transformation of random vectors

1 Transformation of random variables

Suppose that X_1 and X_2 are jointly continuous random variables such that

$$Y_1 = g(X_1, X_2), \qquad Y_2 = h(X_1, X_2).$$

Let Y_1, Y_2 be jointly continuous random variables with density function $f_{Y_1,Y_2}(y_1, y_2)$.

Remark 1. This condition need not always be true.

Now, given Y_1 and Y_2 are random variables we are interested in the probability of the event

$$A \triangleq \{\omega : y_1 < Y_1(\omega) \leqslant y_1 + dy_1, y_2 < Y_2(\omega) \leqslant y_2 + dy_2\}.$$

For continuous random variables Y_1, Y_2 with the joint density $f_{Y_1,Y_2}(y_1, y_2)$, we can write for infinitesimally small dy_1, dy_2

$$P\{\omega : y_1 < Y_1(\omega) \leqslant y_1 + dy_1, y_2 < Y_2(\omega) \leqslant y_2 + dy_2\} \approx f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2.$$

Let us define a mapping, $T = (g, h)$ which yields,

$$y_1 = g(x_1, x_2), \qquad y_2 = h(x_1, x_2).$$

If T is a one-to-one mapping such that $T(x_1, x_2) = (y_1, y_2)$, then there exists an inverse mapping T^{-1} such that

$$(x_1, x_2) = T^{-1}(y_1, y_2).$$

Now consider an elemental area $dA(x_1, x_2)$ in the vicinity of (x_1, x_2). Then, we have the mapping $dA(x_1, x_2) \xrightarrow{\text{T}} dA(y_1, y_2)$ and the inverse mapping $dA(y_1, y_2) \xrightarrow{T^{-1}} dA(x_1, x_2)$. It turns out that,

$$dA(x_1, x_2) = |J(y_1, y_2)| dy_2 dy_1,$$

where, $|J(y_1, y_2)|$ is the Jacobian matrix of T^{-1} given by

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

As T is a one-to-one map, we see that,

$$\{\omega : y_1 < Y_1(\omega) \leqslant y_1 + dy_1, y_2 < Y_2(\omega) \leqslant y_2 + dy_2\} \equiv \{\omega : (X_1(\omega), X_2(\omega)) \in dA(x_1, x_2)\}.$$

As these events are equal, their probabilities must also be equal. Thus, we have,

$$f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2 \approx f_{X_1,X_2}(x_1, x_2)|J(y_1, y_2)|dy_1dy_2 \;\; as \;\; dy_1, dy_2 \to 0.$$

Finally, we have,

$$f_{Y_1,Y_2}(y_1, y_2) \approx f_{X_1,x_2}(x_1(y_1, y_2), x_2(y_1, y_2))|J(y_1, y_2)|$$

Example 1.1 (Sum of random variables). Suppose that X_1 and X_2 are jointly continuous random variables and $Y_1 = X_1 + X_2$. Let us compute $f_{Y_1}(y_1)$ using the above relation. Let us define $Y_2 = X_2$ so that $|J(y_1, y_2)| = 1$. This implies, $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)$. Thus, we may compute the marginal density of Y_1 as,

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) dy_2 = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2) dy_2.$$

Special Case : Suppose X_1 and X_2 are independent. Then,

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1}(y_1 - y_2) f_{X_2}(y_2) dy_2 = (f_{X_1} * f_{X_2})(y_1),$$

where $*$ represents convolution.

Now, let us try to compute the above density $f_{Y_1} y_1$ in a more direct manner. We know,

$$f_{Y_1}(y_1) = \frac{dF_{Y_1}(y_1)}{dy_1}.$$

where, $F_{Y_1}(y_1) = P\{Y_1 \leqslant y_1\} = P\{X_1 + X_2 \leqslant y_1\}$. Representing this probability as the area under the joint density function, we have,

$$F_{Y_1}(y_1) = \int_{\infty}^{\infty} dx_1 \int_{-\infty}^{y_1 - x_1} f_{X_1,X_2}(x_1, x_2) dx_2.$$

Now, by applying change of variable $(x_1, t) = (x_1, x_1 + x_2)$ and changing the order of integration, we see that,

$$F_{y_1}(y_1) = \int_{-\infty}^{y_1} dt \int_{-\infty}^{\infty} dx_1 f_{X_1,X_2}(x_1, t - x_1).$$

Finally, we get the expected result

$$f_{Y_1}(y_1) = \frac{dF_{Y_1}(y_1)}{dy_1} = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y_1 - x_1) dx_1.$$

2 Characteristic function

Suppose that X is a continuous random variable with density $f_X(x)$. Then, recall that the characteristic function is given by

$$\mathbb{E}[e^{juX}] = \int_{-\infty}^{\infty} e^{juX} f_X(x) dx,$$

where, $e^{juX} = \cos uX + j \sin uX$.