Lecture-12: Characteristic function

1 Characteristic function

Definition 1.1. Let $j \in \mathbb{C}$ such that $j^2 = -1$, then the **characteristic function** $\Phi_X : \mathbb{R} \to \mathbb{R}$, of a random variable X is defined as

$$\Phi_X(u) \triangleq \mathbb{E}[e^{juX}], \text{ for all } u \in \mathbb{R}.$$

Remark 1. Following statements are true for characteristic functions.

1. Since $e^{j\theta} = \cos\theta + j\sin\theta$, the characteristic function can be equivalently written as
$$\Phi_X(u) = \mathbb{E}[\cos uX] + j\mathbb{E}[\sin uX].$$
2. Suppose that $X : \Omega \to \mathcal{X}$ is a discrete random variable. Then
$$\Phi_X(u) = \mathbb{E}[e^{juX}] = \sum_{x \in \mathcal{X}: P_X(x) > 0} e^{jux} P_X(x)$$
3. Suppose that $X : \Omega \to \mathbb{R}$ is a continuous random variable with density function $f_X : \mathbb{R} \to \mathbb{R}_+$. Then,
$$\Phi_X(u) = \int_{-\infty}^{\infty} e^{juX} f_X(x) dx.$$
4. The characteristic function $\Phi_X(u)$ is always finite, since
$$|\Phi_X(u)| = \left| \int_{-\infty}^{\infty} e^{juX} f_X(x) dx \right| \leqslant \int_{-\infty}^{\infty} \left| e^{juX} f_X(x) \right| dx = \int_{-\infty}^{\infty} f_X(x) dx = 1.$$

Lemma 1.2. *If $\mathbb{E}\,|X|^k$ is finite, then $\mathbb{E}\,|X|^i$ is finite for $i \in [k]$.*

Proof. It can be easily seen when $X : \Omega \to \mathcal{X} \subset \mathbb{R}_+$ is a non-negative discrete random variable with probability mass function $P_X : \mathcal{X} \to [0,1]$. We partition the alphabet $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ into disjoint subsets where

$$\mathcal{X}_0 \triangleq \{x \in \mathcal{X} : x \in [0,1]\}.$$

Then, we can write

$$\mathbb{E}[X^i] = \sum_{x \in \mathcal{X}} x^i P_X(x) = \sum_{x \in \mathcal{X}_0} x^i P_X(x) + \sum_{x \in \mathcal{X}_1} x^i P_X(x) \leqslant 1 + \sum_{x \in \mathcal{X}_1} x^k P_X(x) < \infty.$$

Theorem 1.3. *If $\mathbb{E}\,|X|^k$ is finite for some integer $k \geqslant 1$, then $\Phi_X^{(i)}(u)$ for $i = [k]$ are all finite and continuous functions of u. Further, $\Phi_X^{(i)}(0) = j^k \mathbb{E}[X^i]$ for all $i \in [k]$.*

Proof. Let us differentiate the characteristic function $\Phi_X(u)$, with respect to u, to write

$$\Phi_X'(u) = \frac{d\Phi_X(u)}{du} = \frac{d}{du}\left(\int_{-\infty}^{\infty} e^{juX} f_X(x) dx\right).$$

Exchanging derivative and the integration (which can be done since e^{jux} is a bounded function with all derivatives), and evaluating the derivative at $u = 0$, we get

$$\Phi_X'(0) = \int_{-\infty}^{\infty} \frac{de^{juX}}{du}\Big|_{u=0} f_X(x) dx = \int_{-\infty}^{\infty} (jxe^{juX})\Big|_{u=0} f_X(x) dx = j\mathbb{E}[X].$$

It turns out that $\Phi_X'(0) = j\mathbb{E}[X]$, when both L.H.S and R.H.S are finite. Similarly, $\Phi_X^{(k)}(0) = j^k \mathbb{E}[X^k]$, when both L.H.S and R.H.S are finite.

2 Moment Generating Function

Definition 2.1. For a random variable X, the **moment generating function** denoted by $M_X : \mathbb{R} \to \mathbb{R}$, is defined by

$$M_X(t) \triangleq \mathbb{E}[e^{tX}], \text{ for all } t \in \mathbb{R}.$$

Lemma 2.2. *When $M_X(t)$ is finite for some $t \in \mathbb{R}$ and random variable X, we have*

$$M_X(t) = \mathbb{E}[e^{tX}] = 1 + \sum_{n\in\mathbb{N}} \frac{t^n}{n!}\mathbb{E}[X^n].$$

Proof. From the Taylor series expansion of e^θ around $\theta = 0$, we get $e^\theta = 1 + \sum_{n\in\mathbb{N}} \frac{\theta^n}{n!}$. Therefore, taking $\theta = tX$, we can write

$$e^{tX} = 1 + \sum_{n\in\mathbb{N}} \frac{t^n}{n!}X^n.$$

Taking expectation on both sides, the result follows from the linearity of expectation, when both sides have finite expectation.

Definition 2.3. For a discrete random variable $X : \Omega \to \mathcal{X}$ with probability mass function $P_X : \mathcal{X} \to [0,1]$, the **probability generating function** $\Psi_X : \mathbb{C} \to \mathbb{C}$ is defined by

$$\Psi_X(z) = \mathbb{E}[z^X] = \sum_{x\in\mathcal{X}} z^x P_X(x), \; z \in \mathbb{C}.$$

Lemma 2.4. *The absolute value of the probability generating function evaluated at $z \in \mathbb{C}$ with $|z| \leqslant 1$ for a positive simple random variable X, is upper bounded by unity.*

Proof. Let $z \in \mathbb{C}$ with $|z| \leqslant 1$. Let $P_X : \mathcal{X} \to [0,1]$ be the probability mass function of the positive simple random variable X. Since any realization $x \in \mathcal{X}$ of random variable X is positive, we can write

$$|\Psi_X(z)| = \left|\sum_{x\in\mathcal{X}} z^x P_X(x)\right| \leqslant \sum_{x\in\mathcal{X}} |z|^x P_X(x) \leqslant \sum_{x\in\mathcal{X}} P_X(x) = 1.$$

Definition 2.5. For a positive random variable X, the k-th order factorial moment of random variable X is defined as

$$\mathbb{E}\left[\prod_{i=0}^{k-1}(X-i)\right] = \mathbb{E}[X(X-1)(X-2)\dots(X-k+1)].$$

Theorem 2.6. *For a positive simple random variable X, the k-th derivative of probability generating function evaluated at $z = 1$ is the k-th order factorial moment of X. That is,*

$$\Psi_X^{(k)}(1) = \mathbb{E}[X(X-1)(X-2)\dots(X-k+1)].$$

Proof. It follows from the interchange of derivative and expectation.

Remark 2. Moments can be recovered from kth order factorial moments. For example,

$$\mathbb{E}[X] = \Psi_X'(1), \qquad \mathbb{E}[X^2] = \Psi_X^{(2)}(1) + \Psi_X'(1).$$