Lecture-18: Random Processes

1 Introduction

Definition 1.1 (Random process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. For an arbitrary index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, a **random process** is a measurable map $X : \Omega \to \mathcal{X}^T$. That is, for each outcome $\omega \in \Omega$, we have a function $X(\omega) : T \mapsto \mathcal{X}$ called the **sample path** or the **sample function** of the process X, also written as

$$X(\omega) \triangleq (X_t(\omega) \in \mathcal{X} : t \in T).$$

1.1 Classification

State space $\mathcal{X}$ can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set T is countable, the stochastic process is called **discrete**-time stochastic process or random sequence. When the index set T is uncountable, it is called **continuous**-time stochastic process. The index set T doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When $T = \mathbb{R}^n \times [0, \infty)$, stochastic process X is a spatio-temporal process.

Example 1.2. We list some examples of each such stochastic process.

i. Discrete random sequence: brand switching, discrete time queues, number of people at bank each day.

ii. Continuous random sequence: stock prices, currency exchange rates, waiting time in queue of nth arrival, workload at arrivals in time sharing computer systems.

iii. Discrete random process: counting processes, population sampled at birth-death instants, number of people in queues.

iv. Continuous random process: water level in a dam, waiting time till service in a queue, location of a mobile node in a network.

1.2 Measurability

For any finite subset $S \subseteq T$ and real vector $x \in \mathbb{R}^T$ such that $x_t = \infty$ for any $t \notin S$, we define a set $A(x) = \{y \in \mathbb{R}^T : y_t \leqslant x_t\} = \bigtimes_{t \in T}(-\infty, x_t]$. Then, the measurability of the random process X implies that for any such set $A(x)$, we have

$$\{\omega \in \Omega : X_t(\omega) \leqslant x_t, t \in T\} = \cap_{t \in T} X_s^{-1}(-\infty, x_t] = X^{-1} \bigtimes_{t \in T}(-\infty, x_t] = X^{-1}(A(x)) \in \mathcal{F}.$$

Remark 1. Realization of random process at each $t \in T$, is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $X_t : \Omega \to \mathcal{X}$. This follows from the fact that for any $t \in T$ and $x_t \in \mathbb{R}$, we can take Boreal measurable sets $\bigtimes(-\infty, x_t] \bigtimes_{s \neq t} \mathbb{R}$. Then, $X^{-1}(A(y)) = X_t^{-1}(-\infty, x_t] \in \mathcal{F}$.

Remark 2. The random process X can be thought of as a collection of random variables $X = (X_t \in \mathcal{X}^\Omega : t \in T)$ or an ensemble of sample paths $X = (X(\omega) \in \mathcal{X}^T : \omega \in \Omega)$. Recall that $\mathcal{X}^T$ is set of all functions from the index set T to state space $\mathcal{X}$.

Example 1.3 (Bernoulli sequence). Let index set $T = \mathbb{N} = \{1, 2, \dots\}$ and the sample space be the collection of infinite bi-variate sequences of successes (S) and failures (F) defined by $\Omega = \{S, F\}^{\mathbb{N}}$. An outcome $\omega \in \Omega$ is an infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ such that $\omega_n \in \{S, F\}$ for each $n \in \mathbb{N}$. We define the random process $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ such that $X(\omega) = (\mathbb{1}_{\{S\}}(\omega_1), \mathbb{1}_{\{S\}}(\omega_2), \dots)$. That is, we have

$$X_n(\omega) = \mathbb{1}_{\{S\}}(\omega_n), \qquad X(\omega) = (\mathbb{1}_{\{S\}}(\omega_n) : n \in \mathbb{N}).$$

Hence, we can write the process as collection of random variables $X = (X_n \in \{0, 1\}^{\Omega} : n \in \mathbb{N})$ or the collection of sample paths $X = (X(\omega) \in \{0, 1\}^{\mathbb{N}} : \omega \in \Omega)$.

1.3 Distribution

To define a measure on a random process, we can either put a measure on sample paths, or equip the collection of random variables with a joint measure. We are interested in identifying the joint distribution $F : \mathbb{R}^T \to [0, 1]$. To this end, for any $x \in \mathbb{R}^T$we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t \in T} \{\omega \in \Omega : X_t(\omega) \leqslant x_t\}\right) = P(\bigcap_{t \in T} X_t^{-1}(-\infty, x_t]) = P \circ X^{-1} \bigtimes_{t \in T} (-\infty, x_t].$$

However, even for a simple independent process with countably infinite T, any function of the above form would be zero if x_t is finite for all $t \in T$. Therefore, we only look at the values of $F(x)$ when $x_t \in \mathbb{R}$ for indices t in a finite set S and $x_t = \infty$ for all $t \notin S$. That is, for any finite set $S \subseteq T$, we focus on the product sets of the form

$$A(x) \triangleq \bigtimes_{s \in S} (-\infty, x_s] \bigtimes_{s \notin S} \mathbb{R},$$

where $x \in \mathcal{X}^T$ and $x_t = \infty$ for $t \notin S$. Recall that by definition of measurability, $X^{-1}(A(x)) \in \mathcal{F}$, and hence $P \circ X^{-1} A(x)$ is well defined.

Definition 1.4 (Finite dimensional distribution). We can define a **finite dimensional distribution** for any finite set $S \subseteq T$ and $x_S = \{x_s \in \mathbb{R} : s \in S\}$,

$$F_{X_S}(x_S) \triangleq P\left(\bigcap_{s \in S} \{\omega \in \Omega : X_s(\omega) \leqslant x_s\}\right) = P(\bigcap_{s \in S} X_s^{-1}(-\infty, x_s]).$$

Set of all finite dimensional distributions of the stochastic process $X = (X_t \in \mathcal{X}^{\Omega} : t \in T)$ characterizes its distribution completely. Simpler characterizations of a stochastic process X are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E} X_t, \qquad R_X(t, s) \triangleq \mathbb{E} X_t X_s, \qquad C_X(t, s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

Example 1.5. Some examples of simple stochastic processes.

i_ $X_t = A \cos 2\pi t$, where A is random. The finite dimensional distribution is given by

$$F_{X_S}(x) = P\left(\{A \cos 2\pi s \leq x_s, s \in S\}\right).$$

The moments are given by

$$m_X(t) = (\mathbb{E} A) \cos 2\pi t, \quad R_X(t, s) = (\mathbb{E} A^2) \cos 2\pi t \cos 2\pi s, \quad C_X(t, s) = \mathrm{Var}(A) \cos 2\pi t \cos 2\pi s.$$

ii. $X_t = \cos(2\pi t + \Theta)$, where Θ is random and uniformly distributed between $(-\pi, \pi]$. The finite dimensional distribution is given by

$$F_{X_S}(x) = P\left(\{\cos(2\pi s + \Theta) \leq x_s, s \in S\}\right).$$

The moments are given by

$$m_X = 0, \qquad R_X(t,s) = \frac{1}{2}\cos 2\pi(t-s), \qquad C_X(t,s) = R_X(t,s).$$

iii. $X_n = U^n$ for $n \in \mathbb{N}$, where U is uniformly distributed in the open interval $(0,1)$.

iv. $Z_t = At + B$ where A and B are independent random variables.

1.4 Independence

Recall, given the probability space $(\Omega, \mathcal{F}, P)$, two events $A, B \in \mathcal{F}$ are **independent events** if

$$P(A \cap B) = P(A)P(B).$$

Random variables X, Y defined on the above probability space, are **independent random variables** if for all $x, y \in \mathbb{R}$

$$P\{X(\omega) \leqslant x, Y(\omega) \leqslant y\} = P\{X(\omega) \leqslant x\}P\{Y(\omega) \leqslant y\}.$$

A stochastic process X is said to be **independent** if for all finite subsets $S \subseteq T$, the finite collection of events $\{\{X_s \leqslant x_s\} : s \in S\}$ are independent. That is, we have

$$F_{X_S}(x_S) = P(\cap_{s \in S}\{X_s \leqslant x_s\}) = \prod_{s \in S} P\{X_s \leqslant x_s\} = \prod_{s \in S} F_{X_s}(x_s).$$

Two stochastic process X, Y for the common index set T are **independent random processes** if for all finite subsets $I, J \subseteq T$, the following events $\{X_i \leqslant x_i, i \in I\}$ and $\{Y_j \leqslant y_j, j \in J\}$ are independent. That is,

$$F_{X_I, X_J}(x_I, x_J) \triangleq P\left(\{X_i \leqslant x_i, i \in I\} \cap \{Y_j \leqslant y_j, j \in J\}\right) = P\left(\cap_{i \in I}\{X_i \leqslant x_i\}\right) P\left(\cap_{j \in J}\{Y_j \leqslant y_j\}\right) = F_{X_I}(x_I) F_{X_J}(x_J).$$

Example 1.6 (Bernoulli sequence). Let the Bernoulli sequence X defined in Example 1.3 be independent and identically distributed with $P\{X_n = 1\} = p \in (0,1)$. For any sequence $x \in \{0,1\}^{\mathbb{N}}$, we have $P\{X = x\} = 0$. Let $q \triangleq (1-p)$, then the probability of observing m heads and r tails is given by $p^m q^r$. We can easily compute the mean, the auto-correlation, and the auto-covariance functions for the independent Bernoulli process defined in Example 1.6 as

$$m_X(n) = \mathbb{E}X_n = p, \qquad R_X(m,n) = \mathbb{E}X_m X_n = \mathbb{E}X_m \mathbb{E}X_n = p^2, \qquad C_x(m,n) = 0.$$