# Lecture-19: Tractable Random Processes

## 1 Examples of Tractable Stochastic Processes

In general, it is very difficult to characterize a stochastic process completely in terms of its finite dimensional distribution. However, we have listed few analytically tractable examples below, where we can completely characterize the stochastic process.

We will consider the probability space $(\Omega, \mathcal{F}, P)$, and a random process $X : \Omega \to \mathcal{X}^T$ for index set $T$ and state space $\mathcal{X} \subseteq \mathbb{R}$.

### 1.1 Independent and identically distributed (IID) processes

**Definition 1.1 (IID process).** A random process $X : \Omega \to \mathcal{X}^T$ is an **independent and identically distributed** (*iid*) random process with the common distribution $F(x)$, if for any finite $S \subseteq T$ and a real vector $x_S \in \mathbb{R}^S$ we can write the finite dimensional distribution for this process as

$$F_S(x_S) = P\left(\{X_s(\omega) \leqslant x_s, s \in S\}\right) = \prod_{s \in S} F(x_s).$$

*Remark* 1. It's easy to verify that the first and the second moments are independent of time indices. That is, if $0 \in T$ then $X_t = X_0$ in distribution, and we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t,s) = (\mathbb{E}X_0^2)\mathbb{1}_{\{t=s\}} + m_X^2 \mathbb{1}_{\{t \neq s\}}, \qquad C_X(T,s) = \mathrm{Var}(X_0)\mathbb{1}_{\{t=s\}}.$$

### 1.2 Stationary processes

**Definition 1.2 (Stationary process).** We consider the index set $T \subseteq \mathbb{R}$. A stochastic process $X : \Omega \to \mathcal{X}^T$ is **stationary** if all finite dimensional distributions are shift invariant. That is, for any finite $S \subseteq T$ and $t > 0$, we have

$$F_S(x_S) = P(\{X_s(\omega) \leqslant x_s, s \in S\}) = P(\{X_{s+t}(\omega) \leqslant x_s, s \in S\}) = F_{t+S}(x_S).$$

*Remark* 2. That is, for any finite $n \in \mathbb{N}$ and $t > 0$, the random vectors $(X_{s_1}, \dots, X_{s_n})$ and $(X_{s_1+t}, \dots, X_{s_1+t})$ have the identical joint distribution for all $s_1 \leqslant \dots \leqslant s_n$.

**Lemma 1.3.** *Any* i.i.d. *process with index set $T \subseteq \mathbb{R}$ is stationary.*

*Proof.* Let $X : \Omega \to \mathcal{X}^T$ be an *i.i.d.* random process, where $T \subseteq \mathbb{R}$. Then, for any finite index subset $S \subseteq T, t \in T$ and $x_S \in \mathbb{R}^S$, we can write

$$F_S(x_S) = P(\{X_s \leqslant x_s, s \in S\}) = \prod_{s \in S} P\{X_s \leqslant x_s\} = \prod_{s \in S} P\{X_{s+t} \leqslant x_s\} = P\{X_{t+u} \leqslant x_u, u \in S\} = F_{t+S}(x_S).$$

First equality follows from the definition, the second from the independence of process $X$, the third from the identical distribution for the process $X$. In particular, we have shown that process $X$ is also stationary. ☐

*Remark* 3. For a stationary stochastic process, all the existing moments are shift invariant when they exist.

**Definition 1.4.** A **second order** stochastic process $X$ has finite auto-correlation $R_X(t,t) < \infty$ for all indices $t \in T$.

*Remark* 4. This implies $R_X(t_1,t_2) < \infty$ by Cauchy-Schwartz inequality, and hence the mean, auto-correlation, and the auto-covariance functions are well defined and finite.

*Remark* 5. For a stationary process $X$, we have $X_t = X_0$ and $(X_t, X_s) = (X_{t-s}, X_0)$ in distribution. Therefore, for a second order stationary process $X$, we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t,s) = R_X(t-s,0) = \mathbb{E}X_{t-s}X_0, \qquad C_X(t-s,0) = R_X(t-s,0) - m_X^2.$$

**Definition 1.5.** A random process $X$ is **wide sense stationary** if

1. $m_X(t) = m_X(t+s)$ for all $s,t \in T$, and
2. $R_X(t,s) = R_x(t+u,s+u)$ for all $s,t,u \in T$.

*Remark* 6. It follows that a second order stationary stochastic process $X$, is wide sense stationary. A second order wide sense stationary process is not necessarily stationary. We can similarly define join stationarity and joint wide sense stationarity for two stochastic processes $X$ and $Y$.

**Example 1.6 (Gaussian process).** Let $X : \Omega \to \mathbb{R}^{\mathbb{R}}$ be a zero-mean continuous-time Gaussian process, defined by its finite dimensional distributions. In particular, for any finite $S \subset \mathbb{R}$, column vector $x_S \in \mathbb{R}^S$, and the covariance matrix $C_S \triangleq \mathbb{E}x_S x_S^T$, the finite-dimensional density is given by

$$f_S(x_S) = \frac{1}{(2\pi)^{|S|/2}\sqrt{\det(C_S)}} \exp\left(-\frac{1}{2}x_S^T C_S^{-1} x_S\right).$$

**Theorem 1.7.** *A wide sense stationary Gaussian process is stationary.*

*Proof.* For Gaussian random processes, first and the second moment suffice to get any finite dimensional distribution. Let $X$ be a wide sense stationary Gaussian process and let $S \subseteq \mathbb{R}$ be finite. From the wide sense stationarity of $X$, we have $\mathbb{E}X_S = 0$ and

$$\mathbb{E}X_s X_u = C_{s-u}, \quad \text{for all } s,u \in S.$$

This means that $C_S = C_{t+S}$, and the result follows.

## 1.3 Markov processes

A stochastic process $X$ is **Markov** if conditioned on the present state, future is independent of the past. We denote the history of the process until time $t$ as $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$. That is, for any ordered index set $T$ containing any two indices $u > t$, we have

$$P(\{X_u \leqslant x_u\} \mid \mathcal{F}_t) = P(\{X_u \leqslant x_u\} \mid \sigma(X_t)).$$

The range of the process is called the **state space**. We next re-write the Markov property more explicitly for the process $X$. For all $x,y \in \mathcal{X}$, finite set $S \subseteq T$ such that $\max S < t < u$, and $H_S = \cap_{s\in S}\{X_s \leqslant x_s\} \in \mathcal{F}_t$, we have

$$P(\{X_u \leqslant y\} \mid H_S \cap \{X_t \leqslant x\}) = P(\{X_u \leqslant y\} \mid \{X_t \leqslant x\}).$$

When the state space $\mathcal{X}$ is countable, we can write $H_S = \cap_{s\in S}\{X_s = x_s\}$ and the Markov property can be written as

$$P(\{X_u = y\} \mid H_S \cap \{X_t = x\}) = P(\{X_u = x_u\} \mid \{X_t = x\}).$$

In addition, when the index set is countable, i.e. $T = \mathbb{Z}_+$, then we can take past as $S = \{0, \dots, n-1\}$, present as instant $n$, and the future as $n+1$. Then, the Markov property can be written as

$$P(\{X_{n+1} = y\} \mid H_{n-1} \cap \{X_n = x\}) = P(\{X_{n+1} = y\} \mid \{X_n = x\}),$$

for all $n \in \mathbb{Z}_+, x,y \in \mathcal{X}$. We will study this process in detail in coming lectures.

## 1.4 Lévy processes

A right continuous with left limits stochastic process $X = (X_t \in \mathbb{R} : t \in T \subseteq \mathbb{R}_+)$ with $X_0 = 0$ almost surely, is a **Lévy process** if the following conditions hold.

(L1) The increments are independent. For any instants $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L2) The increments are stationary. For any instants $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and time-difference $s > 0$, the random vectors $(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}})$ and $(X_{s+t_2} - X_{s+t_1}, X_{s+t_3} - X_{s+t_2}, \dots, X_{s+t_n} - X_{s+t_{n-1}})$ are equal in distribution.

(L3) Continuous in probability. For any $\epsilon > 0$ and $t \geq 0$ it holds that $\lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.

**Example 1.8.** Two examples of Lévy processes are Poisson process and Wiener process. The distribution of Poisson process at time $t$ is Poisson with rate $\lambda t$ and the distribution of Wiener process at time $t$ is zero mean Gaussian with variance $t$.