# Lecture-22: DTMC: Strong Markov Property

## 1 Introduction

We are interested in generalizing the Markov property to any random times. For a DTMC $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, let $T : \Omega \to \mathbb{N}$ be an integer random variable, and we are interested in knowing whether for any historical event $H_{T-1} = \cap_{n=0}^{T-1} \{X_n = x_n\}$ and any state $x, y \in \mathcal{X}$, we have

$$P(\{X_{T+1} = y\} \mid H_{T-1} \cap \{X_T = x\}) = p_{xy}.$$

**Example 1.1 (Two-state DTMC).** For the two state Markov chain $X \in \{0,1\}^{\mathbb{Z}_+}$ such that $P_0\{X_1 = 1\} = q$ and $P_1\{X_1 = 0\} = p$ for $p, q \in [0,1]$. Let $T : \Omega \to \mathbb{N}$ be an integer random variable defined as

$$T \triangleq \sup\{n \in \mathbb{N} : X_i = 0, \text{ for all } i \leqslant n\}.$$

That is, $\{T = n\} = \{X_1 = 0, \dots, X_n = 0, X_{n+1} = 1\}$. Hence, for the historical event $H_{T-1} = \{X_1 = \dots, X_{T-1} = 0\}$, the conditional probability $P(\{X_{T+1} = 1\} \mid H_{T-1} \cap \{X_T = 0\}) = 1$, and not equal to $q$.

### 1.1 Stopping Time

**Definition 1.2.** For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, a random variable $T : \Omega \to \mathbb{N}$ is called a **stopping time** with respect to this random sequence $X$ if

(i) the stopping time is finite almost surely, i.e. $P\{T < \infty\} = 1$, and

(ii) The event $\{T = n\} \in \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.

That is, given the history of the process until time $n$, we can tell whether the stopping time is $n$ or not. In particular, $P(\{T = n\} \mid \sigma(X_1, \dots, X_n))$ is either one or zero.

**Example 1.3 (Simple random walk).** Let $X$ be an integer random walk starting at origin and with *i.i.d.* step-size sequence $Z$ taking values in $\{-1, 1\}$ with probability of positive step being $P\{Z_n = 1\} = \frac{1}{2}$. We define a random variable

$$T \triangleq \inf\{n \in \mathbb{N} : X_n = 0\}.$$

We first examine the event $\{T = n\}$, and write

$$\{T = n\} = \{X_1 \neq 0, X_2 \neq 0, \dots, X_{n-1} \neq 0, X_n = 0\} \in \sigma(X_0, \dots, X_n).$$

Then, we compute the following probability

$$P\{T < \infty\} = \sum_{n \in \mathbb{N}} P\{T = 2n\} = \sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^{2n} C_n = 1.$$

**Example 1.4 (Two-state DTMC).** Let $X : \Omega \to \{0,1\}^{\mathbb{N}}$ be a two state DTMC with the transition matrix $P$ such that $p_{11} = 1 - p$ and $p_{00} = 1 - q$ for $p, q \in [0,1]$. Then, we can define a random variable

$$T \triangleq \inf\{n \in \mathbb{N} : X_n = 1\}.$$

We first examine the event $\{T = n\}$, and write

$$\{T = n\} = \{X_1 = 0, X_2 = 0, \dots, X_{n-1} = 0, X_n = 1\} \in \sigma(X_0, \dots, X_n).$$

Then, we compute the following probability

$$P_1\{T < \infty\} = \sum_{n \in \mathbb{N}} P_1\{T = n\} = 1 - p + \sum_{n \geqslant 2} p(1-q)^{n-2} q = 1.$$

Similarly, we can compute that $P_0\{T < \infty\} = 1$.

**Lemma 1.5 (Wald's Lemma).** *Consider a random walk $X : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ with* i.i.d. *step-sizes $Z : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|Z_1|$. Let $T$ be a finite mean stopping time with respect to this random walk. Then,*

$$\mathbb{E}[X_T] = \mathbb{E}[Z_1]\,\mathbb{E}[T].$$

*Proof.* From the independence of step sizes, it follows that $Z_n$ is independent of $\sigma(X_0, X_1, \dots, X_{n-1})$. Since $T$ is a stopping time with respect to random walk $X$, we observe that $\{T \geqslant n\} = \{T > n-1\} \in \sigma(X_0, X_1, \dots, X_{n-1})$, and hence it follows that random variable $Z_n$ and $\mathbb{1}_{\{T \geqslant n\}}$ are independent and $\mathbb{E}[Z_n 1_{\{T \geqslant n\}}] = \mathbb{E}Z_n \mathbb{E}\mathbb{1}_{\{\{T \geqslant n\}\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{T} Z_n = \mathbb{E}\sum_{n \in \mathbb{N}} Z_n 1_{\{T \geqslant n\}} = \sum_{n \in \mathbb{N}} \mathbb{E}Z_n \mathbb{E}\left[1_{\{T \geqslant n\}}\right] = \mathbb{E}Z_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} 1_{\{T \geqslant n\}}\right] = \mathbb{E}[Z_1]\mathbb{E}[T].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it by the application of dominated convergence theorem. □

**Corollary 1.6.** *Consider the stopping time $T_i = \min\{n \in \mathbb{N} : X_n = i\}$ for an integer random walk $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$ with* i.i.d. *steps $Z : \Omega \to \mathbb{Z}^{\mathbb{N}}$. Then, the mean of stopping time $\mathbb{E}T_i = i/\mathbb{E}Z_1$.*

*Proof.* This follows from the Wald's Lemma and the fact that $X_{T_i} = i$. □

## 1.2 Strong Markov property (SMP)

**Definition 1.7.** Let $T$ be an integer valued stopping time with respect to a random sequence $X$. Then for all states $x, y \in \mathcal{X}$ and the event $H_{T-1} = \cap_{n=0}^{T-1}\{X_n = x_n\}$, the process $X$ satisfies the **strong Markov property** if

$$P(\{X_{T+1} = y\} \mid \{X_T = x\} \cap H_{T-1}) = P(\{X_{T+1} = y\} \mid \{X_T = x\}).$$

**Lemma 1.8.** *Homogeneous Markov chains satisfy the strong Markov property.*

*Proof.* Let $X \in \mathcal{X}^{\mathbb{Z}_+}$ be a homogeneous DTMC with transition matrix $P$. We take any historical event $H_{T-1} = \cap_{n=0}^{T-1}\{X_n = x_n\}$, and $x, y \in \mathcal{X}$. Then, from the definition of conditional probability, the law of total probability, and the Markovity of the process $X$, we have

$$\begin{aligned}
P(\{X_{T+1} = y\} \mid H_{T-1} \cap \{X_T = x\}) &= \frac{\sum_{n \in \mathbb{Z}_+} P(\{X_{T+1} = y, X_T = x\} \cap H_{T-1} \cap \{T = n\})}{P(\{X_T = x\} \cap H_{T-1})} \\
&= \sum_{n \in \mathbb{Z}_+} P(\{X_{n+1} = y\} \mid \{X_n = x\} \cap H_{n-1} \cap \{T = n\}) P(\{T = n\} \mid \{X_T = x\} \cap H_{T-1}) \\
&= p_{xy} \sum_{n \in \mathbb{Z}_+} P(\{T = n\} \mid \{X_T = x\} \cap H_{T-1}) = p_{xy}.
\end{aligned}$$

This equality follows from the fact that the event $\{T = n\}$ is completely determined by $(X_0, \dots, X_n)$. □

*Remark* 1. We have already seen an example where SMP doesn't hold.

i. As an exercise, if we try to use the Markov property on arbitrary random variable $T$, the SMP may not hold. For example, define a non-stopping time $T$ for $y \in \mathcal{X}$

$$T = \inf\{n \in \mathbb{Z}_+ : X_{n+1} = y\}.$$

In this case, we have

$$P(\{X_{T+1} = y\} \mid \{X_T = x, \dots, X_0 = x_0\}) = 1_{\{p_{xy}>0\}} \neq P(\{X_1 = y\} \mid \{X_0 = x\}) = p_{xy}.$$

ii. A useful application of the strong Markov property is as follows. Let $x_0 \in \mathcal{X}$ be a fixed state and $\tau_0 = 0$. Let $\tau_n$ denote the stopping times at which the Markov chain visits $x_0$ for the $n$th time. That is,

$$\tau_n \triangleq \inf\{n > \tau_{n-1} : X_n = x_0\}.$$

Then $(X_{\tau_n+m} \in \mathcal{X}^\Omega : m \in \mathbb{Z}_+)$ is a stochastic replica of $(X_m \in \mathcal{X}^\Omega : m \in \mathbb{Z}_+)$ with $X_0 = x_0$.