Lecture-23: DTMC: Hitting and Recurrence Times

1 Hitting and Recurrence Times

Definition 1.1. Let $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ be a time-homogeneous Markov chain on state space $\mathcal{X}$ with transition probability matrix P. For each state $y \in \mathcal{X}$, we can define the **first hitting time to this state** y after time $n = 0$, as

$$H_y \triangleq \inf\{n \in \mathbb{N} : X_n = y\}.$$

Remark 1. We observe that $\{H_y = n\} \in \mathcal{F}_n \triangleq \sigma(X_0, \dots, X_n)$.

Definition 1.2. For each $n \in \mathbb{N}$, we can write the **probability of first visit to state** y **at time** n from the initial state x, as

$$f_{xy}^{(n)} \triangleq P(\{H_y = n\} \mid \{X_0 = x\}) = P_x\{H_y = n\}.$$

Definition 1.3. The probability that the Markov chain X hits state y eventually, starting from initial state x is

$$f_{xy} \triangleq P_x\{H_y < \infty\} = P_x(\cup_{n\in\mathbb{N}}\{H_y = n\}) = \sum_{n\in\mathbb{N}} P_x\{H_y = n\} = \sum_{n\in\mathbb{N}} f_{xy}^{(n)}.$$

Remark 2. If $f_{xy} = P_x\{H_y < \infty\} = 1$ for all initial states $x \in \mathcal{X}$, then H_y is a stopping time.

Definition 1.4. The distribution $((f_{xy}^{(n)} : n \in \mathbb{N}), 1 - f_{xy})$ is called the **first passage time distribution** for hitting state y from initial state x. The distribution $((f_{xx}^{(n)} : n \in \mathbb{N}), 1 - f_{xx})$ is called the **first recurrence time distribution** for return to initial state x.

Definition 1.5. A state is called **recurrent** if $f_{xx} = 1$, and is called **transient** if $f_{xx} < 1$. For a recurrent state $x \in \mathcal{X}$, we can define **mean recurrence time** as

$$\mu_{xx} \triangleq \mathbb{E}_x H_x = \sum_{n\in\mathbb{N}} nP_x\{H_x = n\} = \sum_{n\in\mathbb{N}} nf_{xx}^{(n)}.$$

If the mean recurrence time for a recurrent state x is finite then the state x is called positive recurrent, and **null recurrent** otherwise.

Definition 1.6. Let $S_y^{(0)} = 0$ and define $S_y^{(k)}$ to be the k**th hitting time of state** y, defined as

$$S_y^{(k)} \triangleq \inf\left\{n > S_y^{(k-1)} : X_n = y\right\},\ k \in \mathbb{N}.$$

We can also define the k**th excursion time** as $H_y^{(k)} \triangleq S_y^{(k)} - S_y^{(k-1)}$, for all $k \in \mathbb{N}$.

Lemma 1.7. *The sequence of random variables* $\left\{H_y^{(k)} \in \Omega^{\mathbb{N}} : k \geqslant 2\right\}$ *is* i.i.d..

Proof. We will show that $H_y^{(k)}$ and $H_y^{(k+1)}$ are independent for any $k \geqslant 2$, and the rest follows from induction. □

Definition 1.8. For a process $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the number of visits to a state $y \in \mathcal{X}$ in n time steps and its limit as $n \to \infty$ are defined as

$$N_y(n) \triangleq \sum_{k=1}^{n} \mathbb{1}_{\{X_k = y\}}, \qquad N_y \triangleq \lim_n N_y(n) = \sum_{k\in\mathbb{N}} \mathbb{1}_{\{X_k = y\}}.$$

Proposition 1.9. *The total number of visits to a state $y \in \mathcal{X}$ is denoted by $N_y = \sum_{n \in \mathbb{N}} 1_{\{X_n = y\}}$. Then, for each $m \in \mathbb{Z}_+$, we have*

$$P_x\{N_y = m\} = \begin{cases} 1 - f_{xy}, & m = 0, \\ f_{xy} f_{yy}^{m-1}(1 - f_{yy}), & m \in \mathbb{N}. \end{cases}$$

Proof. Conditioned on $X_0 = x$, the first passage time H_y to state y being finite is a Bernoulli random variable with probability f_{xy}. The time of the mth return to the state y is a recurrence time for each $m \in \mathbb{Z}_+$. From strong Markov property, each return to state y is independent of the past. Hence, each return to state y in a finite time is an *iid* Bernoulli random variable with probability f_{yy}. It follows that the number of recurrences to state y is the time for first failure to return. Conditioned on initial state being $X_0 = y$, the distribution of N_y is geometric random variable with failure probability $1 - f_{yy}$. □

Proof. We can write $P_x\left\{N_y = 0\right\} = P_x\left\{H_y = \infty\right\} = 1 - f_{xy}$. For $m \in \mathbb{N}$, we consider $P_x\left\{N_y > m\right\}$. Let $S_y^{(0)} = 0$ and define $S_y^{(k)}$ to be the kth hitting time of state y, defined as $S_y^{(k)} \triangleq \inf\left\{n > S_y^{(k-1)} : X_n = y\right\}$. Then, we define the excursion times as $H_y^{(k)} \triangleq S_y^{(k)} - S_y^{(k-1)}$, and define events $E_k \triangleq \left\{S_y^{(k)} < \infty\right\}$ for all $k \in \mathbb{N}$. Then,

$$P_x\left\{N_y = m\right\} = P_x(\left\{S_y^{(m)} < \infty\right\} \cap \left\{S_y^{m+1} = \infty\right\}) = P_x(E_m \cap E_{m+1}^c).$$

We observe that $(E_k : k \in \mathbb{N})$ is a decreasing sequence of events, and hence $E_m = \cap_{k=1}^m E_k$. Together with this observation and from the definition of conditional probability, we can write

$$P_x\left\{N_y = m\right\} = P_x(E_1 \cap E_2 \cap \cdots \cap E_m \cap E_{m+1}^c) = P_x(E_1)\left(\prod_{k=2}^m P(E_k \mid E_1 \cap \cdots \cap E_{k-1})\right) P(E_{m+1}^c \mid E_1 \cap \cdots \cap E_m).$$

From the definition, we get $P_x(E_1) = P_x\left\{H_y < \infty\right\} = f_{xy}$. We focus on the conditional probability of the following event for $k \in \{2, \dots, m\}$, which equals

$$P_x(E_k \mid E_1 \cap \cdots \cap E_{k-1}) = P_x(E_k \mid E_{k-1}) = P(\left\{H_y^{(k)} < \infty\right\} \mid \left\{X_{S_y^{(k-1)}} = y, S_y^{(k-1)} < \infty, X_0 = x\right\}) = P_y\left\{H_y^{(k)} < \infty\right\} = f_{yy}.$$

Equality in the second line follows from the strong Markov property and the definition of f_{yy}. The result follows from the aggregation of the above equalities. □

Corollary 1.10. *For a homogeneous Markov chain X, we have $P_x\{N_y < \infty\} = 1_{\{f_{yy}<1\}} + (1 - f_{xy})1_{\{f_{yy}=1\}}$.*

Proof. We can write the event $\{N_y < \infty\}$ as disjoint union of events $\{N_y = n\}$, to get

$$P_x\{N_y < \infty\} = \sum_{n \in \mathbb{Z}_+} P_x\{N_y = n\} = 1_{\{f_{yy}<1\}} + (1 - f_{xy})1_{\{f_{yy}=1\}}.$$

□

Remark 3. For a homogeneous Markov chain X, we have $P_x\left\{N_y = \infty\right\} = f_{xy}\mathbb{1}_{\{f_{yy}=1\}}$.

Corollary 1.11. *The mean number of visits to state y, starting from a state x is*

$$\mathbb{E}_x N_y = \begin{cases} \frac{f_{xy}}{1-f_{yy}}, & f_{yy} < 1, \\ \infty, & f_{xy} > 1, f_{yy} = 1, \\ 0, & f_{xy} = 0, f_{yy} = 1. \end{cases}$$

Remark 4. For any $x \in \mathcal{X}$, we have $\mathbb{E}_x N_x = \frac{f_{xx}}{1-f_{xx}}\mathbb{1}_{\{f_{xx}<1\}} + \infty\mathbb{1}_{\{f_{xx}=1\}}$.

Remark 5. In particular, this corollary implies the following consequences.

i. A transient state is visited a finite amount of times almost surely. This follows from Corollary 1.10, since $P_x\left\{N_y < \infty\right\} = 1$ for all transient states $y \in \mathcal{X}$ and any initial state $x \in \mathcal{X}$.

ii. A recurrent state is visited infinitely often almost surely. This also follows from Corollary 1.10, since $P_y\left\{N_y < \infty\right\} = 0$ for all recurrent states $y \in \mathcal{X}$.

iii. In a finite state Markov chain, not all states may be transient.

Proof. To see this, we assume that for a finite state space $\mathcal{X}$, all states $y \in \mathcal{X}$ are transient. Then, we know that N_y is finite almost surely for all states $y \in \mathcal{X}$. It follows that, for any initial state $x \in \mathcal{X}$

$$0 \leqslant P_x\left\{\sum_{y\in\mathcal{X}} N_y = \infty\right\} = P_x(\cup_{y\in\mathcal{X}}\left\{N_y = \infty\right\}) \leqslant \sum_{y\in\mathcal{X}} P_x\left\{N_y = \infty\right\} = 0.$$

It follows that $\sum_{x\in\mathcal{X}} N_x$ is also finite almost surely for all states $y \in \mathcal{X}$ for finite state space $\mathcal{X}$. However, we know that $\sum_{x\in\mathcal{X}} N_x = \sum_{k\in\mathbb{N}}\sum_{x\in\mathcal{X}} 1_{\{X_k=x\}} = \infty$. This leads to a contradiction. $\square$

Proposition 1.12. *A state y is recurrent iff $\sum_{k\in\mathbb{N}} p_{yy}^{(k)} = \infty$, and transient iff $\sum_{k\in\mathbb{N}} p_{yy}^{(k)} < \infty$.*

Proof. For any state $x \in \mathcal{X}$, we can write $p_{xx}^{(k)} = P_x\{X_k = x\} = \mathbb{E}_x \mathbb{1}_{\{X_k=x\}}$. Using monotone convergence theorem to exchange expectation and summation, we obtain

$$\sum_{k\in\mathbb{N}} p_{xx}^{(k)} = \mathbb{E}_x \sum_{k\in\mathbb{N}} \mathbb{1}_{\{X_k=x\}} = \mathbb{E}_x N_x.$$

Thus, $\sum_{k\in\mathbb{N}} p_{xx}^{(k)}$ represents the expected number of returns $\mathbb{E}_x N_x$ to a state x starting from state x, which we know to be finite if the state is transient and infinite if the state is recurrent. $\square$

Corollary 1.13. *For a transient state $y \in \mathcal{X}$, the following limits hold $\lim_{n\to\infty} p_{xy}^{(n)} = 0$, and $\lim_{n\to\infty} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = 0$.*

Proof. For a transient state $y \in \mathcal{X}$ and any state $x \in \mathcal{X}$, we have $\mathbb{E}_x N_y = \sum_{n\in\mathbb{N}} p_{xy}^{(n)} < \infty$. Since the series sum is finite, it implies that the limiting terms in the sequence $\lim_{n\to\infty} p_{xy}^{(n)} = 0$. Further, we can write $\sum_{k=1}^n p_{xy}^{(k)} \leqslant \mathbb{E}_x N_y \leqslant M$ for some $M \in \mathbb{N}$ and hence $\lim_{n\to\infty} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = 0$. $\square$

Claim 1.14. *For any state $y \in \mathcal{X}$, let $(H_y^{(\ell)} : \ell \in \mathbb{N})$ be the sequence of almost surely finite inter-visit times to state y, and $N_y(n) = \sum_{k=1}^n 1_{\{X_k=y\}}$ be the number of visits to state y in n times. Then, $N_y(n) + 1$ is a finite mean stopping time with respect to the sequence $(H_y^{(\ell)} : \ell \in \mathbb{N})$.*

Proof. We first observe that $\left\{N_y(n) + 1 = k\right\}$ can be completely determined by observing $H_y^{(1)}, \dots, H_y^{(k)}$. To see this, we notice that

$$\left\{N_y(n) + 1 = k\right\} = \left\{\sum_{\ell=1}^{k-1} H_y^{(\ell)} \leqslant n < \sum_{\ell=1}^{k} H_y^{(\ell)}\right\} \in \sigma(H_y^{(1)}, \dots, H_y^{(k)}).$$

Second, we observe that $N_y(n) + 1 \leqslant n + 1$ and hence has a finite mean for each $n \in \mathbb{N}$. $\square$

We define $N_y(n) \triangleq \sum_{k=1}^n 1_{\{X_k=y\}}$ to be the number of visits to state y in n steps of the Markov process X. Then, $\mathbb{E}_x N_y(n) = \sum_{k=1}^n p_{xy}^{(k)}$.

Theorem 1.15. *Let $x, y \in \mathcal{X}$ be such that $f_{xy} = 1$ and y is recurrent. Then, $\lim_{n\to\infty} \frac{\sum_{k=1}^n p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$.*

Proof. Let $y \in \mathcal{X}$ be recurrent. The proof consists of three parts. In the first two parts, we will show that starting from the state y, we have the limiting empirical average of mean number of visits to state y is $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_y N_y(n) = \frac{1}{\mu_{yy}}$. In the third part, we will show that for any starting state $x \in \mathcal{X}$ such that $f_{xy} = 1$, we have the limiting empirical average of mean number of visits to state y is $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_x N_y(n) = \frac{1}{\mu_{yy}}$.

Lower bound: We observe that $N_y(n) + 1$ is a stopping time with respect to inter-visit times $(H_y^{(\ell)} : \ell \in \mathbb{N})$ from Claim 1.14. Further, we have $\sum_{\ell=1}^{N_y(n)+1} H_y^{(\ell)} > n$. Applying Wald's Lemma to the random sum $\sum_{\ell=1}^{N_y(n)+1} H_y^{(\ell)}$, we get $\mathbb{E}_y(N_y(n)+1)\mu_{yy} > n$. Taking limits, we obtain $\liminf_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{yy}^{(k)}}{n} \geqslant \frac{1}{\mu_{yy}}$.

Upper bound: Consider a counting process with truncated recurrence times $\bar{H}_y^{(\ell)} = M \wedge H_y^{(\ell)}$. It follows that $\bar{N}_y(n) \geqslant N_y(n)$ sample path wise, and $\bar{\mu}_{yy} \triangleq \mathbb{E}_y \bar{H}_y \leqslant \mathbb{E}_y H_y = \mu_{yy}$. Further, we have $\sum_{\ell=1}^{\bar{N}_y(n)+1} \bar{H}_y^{(\ell)} \leqslant n + M$. From Wald's Lemma, we have

$$\mathbb{E}_y(N_y(n)+1)\bar{\mu}_{yy} \leqslant \mathbb{E}_y(\bar{N}_y(n)+1)\bar{\mu}_{yy} \leqslant n + M.$$

Taking limits, we obtain $\limsup_{n\in\mathbb{N}} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} \leqslant \frac{1}{\bar{\mu}_{yy}}$. Letting M grow arbitrarily large, we obtain the upper bound.

Starting from x: Further, we observe that $p_{xy}^{(k)} = \sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)}$. Since $1 = f_{xy} = \sum_{k\in\mathbb{N}} f_{xy}^{(k)}$, we have

$$\sum_{k=1}^{n} p_{xy}^{(k)} = \sum_{k=1}^{n}\sum_{s=0}^{k-1} f_{xy}^{(k-s)} p_{yy}^{(s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k-s=1}^{n-s} f_{xy}^{(k-s)} = \sum_{s=0}^{n-1} p_{yy}^{(s)} - \sum_{s=0}^{n-1} p_{yy}^{(s)} \sum_{k>n-s} f_{xy}^{(k)}.$$

Since the series $\sum_{k\in\mathbb{N}} f_{xy}^{(k)}$ converges, we get $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{yy}^{(k)}}{n}$.

□