Lecture-24: DTMC: Irreducibility and Aperiodicity

1 Communicating classes

For states $x, y \in \mathcal{X}$, it is said that state y is **accessible** from state x if $p_{xy}^{(n)} > 0$ for some $n \in \mathbb{Z}_+$, and denoted by $x \to y$. If two states $x, y \in \mathcal{X}$ are accessible to each other, they are said to **communicate** with each other, denoted by $x \leftrightarrow y$.

1. A set of states that communicate are called a **communicating class**.
2. A communicating class $\mathcal{C}$ is called **closed** if no edges leave this class. That is, for all $x \in \mathcal{C}$ and $y \notin \mathcal{C}$, we have $p_{xy} = 0$.
3. An **open** communicating class is not closed.

Proposition 1.1. *Communication is an equivalence relation.*

Proof. Relation on state space $\mathcal{X}$ is a subset of product of sets $\mathcal{X} \times \mathcal{X}$. Communication is a relation on state space $\mathcal{X}$, as it relates two states $x, y \in \mathcal{X}$. To show equivalence, we have to show reflexivity, symmetry, and transitivity of the relation.

Symmetry: Further, if $x \leftrightarrow y$, then we know that $x \to y$ and $y \to x$ and hence $y \leftrightarrow x$. Hence, the symmetry of the relation follows.

Transitivity: For transitivity, suppose $x \leftrightarrow y$ and $y \leftrightarrow z$. Let $m, n \in \mathbb{Z}_+$ such that $p_{xy}^{(m)} > 0$ and $p_{yz}^{(n)} > 0$. Then by Chapman Kolmogorov equation, we have

$$p_{xz}^{(m+n)} = \sum_{w \in \mathcal{X}} p_{xw}^{(m)} p_{wz}^{(n)} \geqslant p_{xy}^{(m)} p_{yz}^{(n)} > 0.$$

This implies $x \to z$, and using similar arguments one can show that $z \to x$, and the transitivity follows.

Reflexivity: If this relation has a single element, then it is obvious. If not, then for $x \leftrightarrow y$, we have Since $p_{xy}^{(n)} > 0$ and $p_{yx}^{(m)} > 0$ for some $m, n \in \mathbb{Z}_+$. Therefore, $p_{xx}^{(n+m)} \geqslant p_{xy}^{(n)} p_{yx}^{(m)} > 0$ and hence we have $x \leftrightarrow x$, implying the reflexivity of the relation.

□

Hence the communication relation partitions state space $\mathcal{X}$ into equivalence classes. Each equivalence class is called a **communicating class**. A property of states is said to be a **class property** if for each communicating class $\mathcal{C}$, either all states in $\mathcal{C}$ have the property, or none do.

1.1 Irreducibility and periodicity

A Markov chain with a single class is called an **irreducible** Markov chain. That is, for any two states $x, y \in \mathcal{X}$, there exists an integer $n \in \mathbb{N}$ such that $p_{xy}^{(n)} > 0$. That is, any state y can be reached from any state x using transitions of positive probability.

Let $\mathcal{T}(x) \triangleq \left\{ n \in \mathbb{N} : p_{xx}^{(n)} > 0 \right\}$ be the set of times when the chain can possibly return to the initial state x. The **period** of any state $x \in \mathcal{X}$ is defined as

$$d(x) \triangleq \gcd \mathcal{T}(x) = \gcd\{ n \in \mathbb{N} : p_{xx}^{(n)} > 0 \}.$$

We define $d(x) = \infty$, if $p_{xx}^{(n)} = 0$ for all $n \in \mathbb{N}$. A state $x \in \mathcal{X}$ is called **aperiodic** if the period $d(x)$ is 1.

Proposition 1.2. *If $x \leftrightarrow y$, then $d(x) = d(y)$. That is, periodicity is a class property.*

Proof. Let $m, n \in \mathbb{N}$ be such that $p_{xy}^{(m)} p_{yx}^{(n)} > 0$. Suppose $s \in \mathcal{T}(x)$, that is $p_{xx}^{(s)} > 0$. Then

$$p_{yy}^{(n+m)} \geqslant p_{yx}^{(n)} p_{xy}^{(m)} > 0, \qquad p_{yy}^{(n+s+m)} \geqslant p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} > 0.$$

Hence $d(y)|n+m$ and $d(y)|n+s+m$, and hence $d(y)|s$ for any $s \in \mathcal{T}(x)$. In particular, it implies that $d(y)|d(x)$. By symmetrical arguments, we get $d(x)|d(y)$. Hence $d(x) = d(y)$. □

For an irreducible chain, the period of the chain is defined to be the period which is common to all states. An irreducible Markov chain is called **aperiodic** if the single communicating class is aperiodic.

Proposition 1.3. *If the transition matrix P is aperiodic and irreducible, then there is an integer r_0 such that $p_{xy}^{(r)} > 0$ for all $x, y \in \mathcal{X}$ and $r \geqslant r_0$.*

1.2 Transient and recurrent states

Proposition 1.4. *Transience and recurrence are class properties.*

Proof. Let us start with proving recurrence is a class property. Let x be a recurrent state and let $x \leftrightarrow y$. Hence there exist some $m, n > 0$, such that $P_{xy}^{(m)} > 0$ and $p_{yx}^{(n)} > 0$. As a consequence of the recurrence, $\sum_{s \in \mathbb{N}} p_{xx}^{(s)} = \infty$. It follows that y is recurrent by observing

$$\sum_{s \in \mathbb{N}} p_{yy}^{(m+n+s)} \geqslant \sum_{s \in \mathbb{N}} p_{yx}^{(n)} p_{xx}^{(s)} P_{xy}^{(m)} = \infty.$$

Now, if x were transient instead, we conclude that y is also transient by the following observation

$$\sum_{s \in \mathbb{N}} p_{yy}^{(s)} \leqslant \frac{\sum_{s \in \mathbb{N}} p_{xx}^{(m+n+s)}}{p_{yx}^{(n)} P_{xy}^{(m)}} < \infty.$$

□

Corollary 1.5. *If y is recurrent, then for any state x such that $y \to x$, then $x \to y$ and $f_{xy} = 1$.*

Proof. Since $y \to x$, there exists an integer $n \in \mathbb{Z}_+$ such that probability of hitting state x starting from state y in n-steps without revisiting state y is positive. That is,

$$a_{yx}^{(n)} \triangleq P_y \{X_n = x, X_{n-1} \neq y, \ldots, X_1 \neq y\} = P_y \left\{H_y > n, X_n = x\right\} > 0.$$

Suppose $f_{xy} < 1$, then we have

$$1 - f_{yy} = P_y \left\{H_y = \infty\right\} \geqslant P_y \left\{H_y = \infty, X_n = x\right\} = P_x \left\{H_y = \infty\right\} P_y \left\{H_y > n, X_n = x\right\} = a_{yx}^{(n)} (1 - f_{xy}) > 0.$$

This is a contradiction since state y is recurrent. This implies that $f_{xy} = 1$ and hence $x \to y$. □

Corollary 1.6. *Let $x, y \in \mathcal{X}$ be in the same communicating class and the state y is recurrent. Then, $\lim_{n \in \mathbb{N}} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$. Furthermore, if the state y is aperiodic, then $\lim_{n \in \mathbb{N}} p_{xy}^{(n)} = \frac{1}{\mu_{yy}}$.*

Proof. Since y is recurrent and $y \to x$, it follow that $f_{xy} = 1$ from the previous Lemma. From the Theorem 1.7 in previous lecture, it follows that $\lim_{n \in \mathbb{N}} \frac{\sum_{k=1}^{n} p_{xy}^{(k)}}{n} = \frac{1}{\mu_{yy}}$.

Let the period of the state y be d. Then we know that there exists a positive integer r_0 such that for all $n \geqslant r_0$, we have $p_{yy}^{(nd)} > 0$. □

Theorem 1.7. *The states in a communicating class are of one of the following types; all transient, or all null recurrent, or all positive recurrent.*

Proof. It suffices to show that if x, y belong to the same communicating class and y is null recurrent, then x is null recurrent as well. We take $r, s \in \mathbb{N}$, such that $p_{yx}^{(r)} P_{xy}^{(s)} > 0$. It follows that $p_{yy}^{r+\ell+s} \geqslant p_{yx}^{(r)} p_{xx}^{(\ell)} P_{xy}^{(s)}$ for all $\ell \in \mathbb{N}$. Hence, for any $n > r + s$, we have

$$\frac{1}{n} \sum_{k=1}^{n} p_{yy}^{(k)} \geqslant \frac{1}{n} \sum_{k=r+s+1}^{n} p_{yy}^{(k)} \geqslant \left(\frac{n-r-s}{n} \right) \left(\frac{1}{n-r-s} \sum_{\ell=1}^{n-r-s} p_{xx}^{(\ell)} \right) p_{yx}^{(r)} P_{xy}^{(s)}.$$

Since y is null recurrent LHS goes to zero as n increases, which implies $\lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{\ell=1}^{n} p_{xx}^{(\ell)} = 0$. Hence, x is null recurrent as well. $\square$