Lecture-26: Poisson Processes

1 Simple point processes

Consider the d-dimensional Euclidean space $\mathbb{R}^d$, and the collection of Borel measurable subsets $\mathcal{B}(\mathbb{R}^d)$ of the above Euclidean space.

Definition 1.1. A **simple point process** is a random countable collection of distinct points $S : \Omega \to \mathbb{R}^{d^{\mathbb{N}}}$, such that the distance $\|S_n\| \to \infty$ as $n \to \infty$.

Example 1.2 (Simple point process on the half-line). We can simplify this definition for $d = 1$. In $\mathbb{R}_+$, one can order the points $(S_n : n \in \mathbb{N})$ of the point process S, such that $S_1 < S_2 < \cdots < S_n < \ldots$, and $\lim_{n \in \mathbb{N}} S_n = \infty$. The Borel measurable sets for $\mathbb{R}_+$ are generated by the collection of half-open intervals $\{(0, t] : t \in \mathbb{R}_+\}$.

Point processes can model many interesting physical processes.

1. Arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.

Definition 1.3. Corresponding to a point process S, we denote the number of points in a set $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$N(A) = \sum_{n \in \mathbb{N}} 1_{\{S_n \in A\}}, \text{ where we have } N(\emptyset) = 0.$$

Then, $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathbb{R}^d)}$ is called a **counting process** for the point process $S : \Omega \to \mathbb{R}^{d^{\mathbb{N}}}$.

Definition 1.4. A counting process is **simple** if the underlying process is simple.

Remark 1. Let $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathcal{X})}$ be the counting process for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

i. Note that the point process S and the counting process N carry the same information.

ii. The distribution of point process S is completely characterized by the finite dimensional distributions $(N(A_1), \ldots, N(A_k) :$ bounded $A_1, \ldots, A_k \in \mathcal{B})$ for some finite $k \in \mathbb{N}$.

Example 1.5 (Simple point process on the half-line). The number of points in the half-open interval $(0, t]$ is denoted by

$$N(t) \triangleq N((0, t]) = \sum_{n \in \mathbb{N}} 1_{\{S_n \in (0, t]\}}.$$

Since the Borel measurable sets $\mathcal{B}(\mathbb{R}_+)$ are generated by half-open intervals $\{0,t] : t \in \mathbb{R}_+\}$, we denote the counting process by $N : \Omega \to {\mathbb{Z}_+}^{\mathbb{R}_+}$, where $N(t) = N((0,t])$. For $s < t$, the number of points in interval $(s,t]$ is $N((s,t]) = N((0,t]) - N((0,s]) = N(t) - N(s)$.

2 Poisson point process

Definition 2.1. A non-negative integer valued random variable $N \in \mathbb{Z}_+$ is called **Poisson** if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}.$$

Remark 2. It is easy to check that $\mathbb{E}N = \operatorname{Var} N = \lambda$. Furthermore, the moment generating function $M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t-1)}$ exists for all $t \in \mathbb{R}$.

Definition 2.2. For any $k \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+^k$, the **Poisson point process** S **of intensity measure** Λ is defined by its finite dimensional distribution

$$P\left\{N(A_1) = n_1, \dots, N(A_k) = n_k\right\} = \prod_{i=1}^{k}\left(e^{-\Lambda(A_i)\frac{\Lambda(A_i)^{n_i}}{n_i!}}\right),$$

for all bounded mutually disjoint sets $A_1, \dots, A_k \in \mathcal{B}$. If $\Lambda(A) = \lambda\,|A|$, then we call S a **homogeneous Poisson point process** and λ is its intensity.

Remark 3. Recall that $|A| = \int_{x\in A} dx$ is the volume of the set $A \in \mathcal{B}(\mathbb{R}^d)$ and for any such A, the intensity measure of this set is scaled volume

$$\Lambda(A) = \int_{x\in A}\lambda(x)dx,$$

for the intensity density $\lambda : \mathbb{R}^d \to \mathbb{R}_+$. If the intensity density $\lambda(x) = \lambda$ for all $x \in \mathbb{R}^d$, then $\Lambda(A) = \lambda\,|A|$. In particular for partition $A_1, \dots, A_k$ for a set A, we have $\Lambda(A) = \sum_{i=1}^{k}\Lambda(A_i)$.

Definition 2.3. A counting process N has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{B}$, the vector $(N(A_1), \dots, N(A_k))$ is independent. That is,

$$P\bigcap_{i=1}^{k}\{N(A_i) = n_i\} = \prod_{i=1}^{k}P\{N(A_i) = n_i\}.$$

Remark 4. Let $\mathcal{X} = \mathbb{R}^d$, then the process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is a Poisson point process iff

i. the counting process N has complete independence property, and

ii. for each bounded set $A \in \mathcal{B}$, the random variable $N(A)$ is Poisson with parameter $\Lambda(A)$.

In particular, we have $\mathbb{E}N(A) = \Lambda(A)$ for all subsets $A \in \mathcal{B}$.

2.1 Joint conditional distribution of points in a finite window

Proposition 2.4. *For a Poisson point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ and any positive integer $k \in \mathbb{Z}_+$, consider a window $A \in \mathcal{B}(\mathcal{X})$ be a bounded subset, and subsets $(A_1, \dots, A_k)$ that partition this window. Let $n_1, \dots, n_k \in \mathbb{Z}_+$ such that $n_1 + \cdots + n_k = n$, then*

$$P(\{N(A_1) = n_1, \dots, N(A_k) = n_k\} \mid \{N(A) = n\}) = \frac{n!}{n_1! \dots n_k!}\prod_{i=1}^{k}\left(\frac{\Lambda(A_i)}{\Lambda(A)}\right)^{n_i}. \quad (1)$$

Proof. It follows from the definition of joint distribution of $(N(A_1),\dots,N(A_k))$, the fact that $\cap_{i=1}^k \{N(A_i)=n_i\} \subseteq \{N(A)=n\}$, and that the intensity measure add over disjoint sets, i.e. $\Lambda(A)=\sum_{i=1}^k \Lambda(A_i)$. □

Remark 5. Let S be a Poisson point process with intensity measure Λ, and $A_1,\dots,A_k \in \mathcal{B}$ be disjoint bounded subsets such that $A=\cup_{i=1}^k A_i$.

i_ From the disjointness of A_i, we have $N(A)=N(A_1)+\cdots+N(A_k)$, and from the linearity of expectations, we get

$$\Lambda(A)=\mathbb{E}N(A)=\sum_{i=1}^k \mathbb{E}N(A_i)=\sum_{i=1}^k \Lambda(A_i).$$

ii_ Defining $p_i \triangleq \frac{\Lambda(A_i)}{\Lambda(A)}$, we see that $(p_1,\dots,p_k)$ is a probability distribution. We also observe that

$$p_i = P(\{N(A_i)=1\} \mid \{N(A)=1\}) = P(|S\cap A_i|=1 \mid |S\cap A|=1).$$

If we call the point of S in A as S_1, then

$$p_i = P(\{S_1 \in A_i\} \mid \{S_1 \in A\}).$$

In addition, we also observe that

$$p_i^{n_i} = P(\{N(A_i)=n_i\} \mid \{N(A)=n_i\}) = P(|S\cap A_i|=n_i \mid |S\cap A|=n_i).$$

That is, if $S_1,\dots,S_{n_i}$ denote the points of S in A, then

$$p_i^{n_i} = P(\cap_{j=1}^{n_i}\{S_j\in A_i\} \mid \{S_1,\dots,S_{n_i}\in A\}) = \prod_{j=1}^{n_i} P(\{S_j\in A_i\} \mid \{S_j \in A\}).$$

iii_ We can rewrite the equation (1) as a multinomial distribution, where

$$P(\{N(A_1)=n_1,\dots,N(A_k)=n_k\} \mid \{N(A)=n\}) = \binom{n}{n_1,\dots,n_k} p_1^{n_1}\dots p_k^{n_k}.$$

iv_ Let $\mathcal{P}(n_1,\dots,n_k)$ be a collection of all k-partition of $[n]$ such that $|P_i|=n_i$ and $n_1+\cdots+n_k=n$. That is,

$$\mathcal{P}(n_1,\dots,n_k) \triangleq \{(P_1,\dots,P_k) \text{ partition of } [n] : |P_i|=n_i \text{ for all } i\in[k]\}.$$

Then, the multinomial coefficient accounts for number of partitions of n points into sets with $n_1,\dots,n_k$ points. That is,

$$\binom{n}{n_1,\dots,n_k} = |\mathcal{P}(n_1,\dots,n_k)|.$$

v_ We observe that the event $\{N(A_i)=n_i\}=\{|S\cap A_i|=n_i\}$. Hence, we can write

$$\begin{aligned} P(\cap_{i=1}^k\{|S\cap A_i|=n_i\} \mid \{|S\cap A|=n\}) &= \binom{n}{n_1,\dots,n_k} p_1^{n_1}\dots p_k^{n_k} \\ &= \sum_{(E_1,\dots,E_k)\in\mathcal{P}(S\cap A)} \prod_{i=1}^k \prod_{S_j\in E_i} P(\{S_j\in A_i\} \mid \{S_j\in A\}). \end{aligned}$$

vi_ We further observe that $(S\cap A_1,\dots,S\cap A_k)\in\mathcal{P}(n_1,\dots,n_k)$, and hence we can re-write the event

$$\cap_{i=1}^k\{N(A_i)=n_i\} = \cap_{i=1}^k\{|S\cap A_i|=n_i\} = \cup_{(E_1,\dots,E_k)\in\mathcal{P}(n_1,\dots,n_k)}(\cap_{i=1}^k\{S\cap A_i=E_i\}).$$

That is, we can write the conditional probability

$$\begin{aligned} P(\cap_{i=1}^k\{N(A_i)=n_i\} \mid \{N(A)=n\}) &= \sum_{(E_1,\dots,E_k)\in\mathcal{P}(n_1,\dots,n_k)} P(\cap_{i=1}^k\{S\cap A_i=E_i\} \mid \{S\cap A=E\}) \\ &= \sum_{(E_1,\dots,E_k)\in\mathcal{P}(n_1,\dots,n_k)} P(\cap_{i=1}^k \cap_{S_j\in E_i}\{S_j\in A_i\} \mid \{S\cap A=E\}). \end{aligned}$$

vii. Let $S_1,\dots,S_n$ be the n points in $E = S \cap A$. Equating the RHS of the above equation term-wise, we obtain that conditioned on each of these points falling inside the window A, the conditional probability of each point falling in partition A_i is independent of all other points and given by p_i. That is, we have

$$P(\cap_{i=1}^k \cap_{S_j \in E_i} \{S_j \in A_i\} \mid \{S \cap A = E\}) = \prod_{i=1}^k \prod_{S_j \in E_i} P(\{S_j \in A_i\} \mid \{S_j \in A\}) = \prod_{i=1}^k p_i^{n_i} = \prod_{i=1}^k \left(\frac{\Lambda(A_i)}{\Lambda(A)}\right)^{n_i}.$$

It means that given n points in the window A, the location of these points are independently and identically distributed in A according to the distribution $\frac{\Lambda(\cdot)}{\Lambda(A)}$.

viii. If the Poisson process is homogeneous, the distribution is uniform over the window A.

ix. For a Poisson process with intensity measure Λ and any bounded set $A \in \mathcal{B}$, we have $N(A)$ a Poisson random variable with parameter $\Lambda(A)$. Given $N(A)$, the location of all the points in $S \cap A$ are *i.i.d.* with density $\frac{\lambda(x)}{\Lambda(A)}$ for all $x \in A$.

Corollary 2.5. *For a homogeneous Poisson point process on the half-line with ordered set of points* $(S_{(n)} \in \mathbb{R}_+ : n \in \mathbb{N})$, *we can write the conditional density of ordered points* $(S_{(1)},\dots,S_{(k)})$ *given* $N(t) = k$ *as ordered statistics of iid uniformly distributed random variables. Specifically, we have*

$$f_{S_{(1)},\dots,S_{(k)} \mid N(t)=k}(t_1,\dots,t_k) = k! \prod_{i=1}^k \frac{1_{\{t_i \leqslant t\}}}{t}.$$

Proof. Given $N(t) = k$, we can denote the points of the Poisson process in $(0,t]$ by $S_1,\dots,S_k$. From the above remark, we know that $S_1,\dots,S_k$ are *i.i.d.* uniform in $(0,t]$, conditioned on the number of points $N(t) = k$. Hence, we can write

$$F_{S_1,\dots,S_k \mid N(t)=k}(t_1,\dots,t_k) = P(\cap_{i=1}^k \{S_i \in (0,t_i]\} \mid \{N(t) = k\}) = \prod_{i=1}^k P(\{S_i \in (0,t_i]\} \mid \{S_i \in (0,t]\}) = \prod_{i=1}^k \frac{t_i}{t} \mathbb{1}_{\{0 < t_i \leqslant t\}}.$$

For any permutation $\sigma : [k] \to [k]$, the order statistics of $(S_{\sigma(1)},\dots,S_{\sigma(k)})$ are identical. Therefore, we can write the following equality for the events

$$\left\{S_{(i)} \leqslant t_i\right\} = \cup_{\sigma:[k]\to[k] \text{ permutation}} \left\{S_{\sigma(i)} \leqslant t_i\right\}.$$

Hence, the result follows since

$$P(\cap_{i=1}^k \left\{S_{(i)} \in (0,t_i]\right\} \mid \{N(t) = k\}) = \sum_{\sigma:[k]\to[k]} P(\cap_{i=1}^k \left\{S_{\sigma(i)} \in (0,t_i]\right\} \mid \{N(t) = k\}).$$

□