Lecture-27: Properties of Poisson point processes

1 Laplace functional

For a realization of a simple point process S, we observe that $dN(x) = 0$ for all $x \notin S$ and $dN(x) = \delta_x 1_{\{x\in S\}}$. Hence, for any function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\int_{x\in A} f(x)dN(x) = \sum_{i\in\mathbb{N}} f(S_i)1_{\{S_i\in A\}} = \sum_{S_i\in A} f(S_i).$$

Definition 1.1. The **Laplace functional** $\mathcal{L}$ of a point process S and associated counting process N is defined for all non-negative function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathcal{L}_S(f) \triangleq \mathbb{E}\exp\left(-\int_{\mathbb{R}^d} f(x)dN(x)\right).$$

Remark 1. For simple function $f(x) = \sum_{i=1}^k t_i 1_{\{x\in A_i\}}$, we can write the Laplace functional

$$\mathcal{L}_S(f) = \mathbb{E}\exp\left(-\sum_{i=1}^k t_i \int_{A_i} dN(x)\right) = \mathbb{E}\exp\left(-\sum_{i=1}^k t_i N(A_i)\right),$$

as a function of the vector $(t_1, t_2, \dots, t_k)$, a joint Laplace transform of the random vector $(N(A_1), \dots, N(A_k))$. This way, one can compute all finite dimensional distribution of the counting process N.

Proposition 1.2. *The Laplace functional of the Poisson process with intensity measure Λ is*

$$\mathcal{L}_S(f) = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-f(x)})d\Lambda(x)\right).$$

Proof. For a bounded Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$, consider $g(x) = f(x)1_{\{x\in A\}}$. Then,

$$\mathcal{L}_S(g) = \mathbb{E}\exp(-\int_{\mathbb{R}^d} g(x)dN(x)) = \mathbb{E}\exp(-\int_A f(x)dN(x)).$$

Clearly $dN(x) = \delta_x 1_{\{x\in S\}}$ and hence we can write $\mathcal{L}_S(g) = \mathbb{E}\exp\left(-\sum_{S_i\in S\cap A} f(S_i)\right)$. We know that the probability of $N(A) = |S(A)| = n$ points in set A is given by

$$P\{N(A) = n\} = e^{-\Lambda(A)}\frac{\Lambda(A)^n}{n!}.$$

Given there are n points in set A, the density of n point locations are independent and given by

$$f_{S_1,\dots,S_n \mid N(A)=n}(x_1,\dots,x_n) = \prod_{i=1}^n \frac{d\Lambda(x_i)1_{\{x_i\in A\}}}{\Lambda(A)}.$$

Hence, we can write the Laplace functional as

$$\mathcal{L}_S(g) = e^{-\Lambda(A)}\sum_{n\in\mathbb{Z}_+}\frac{\Lambda(A)^n}{n!}\prod_{i=1}^n\int_A e^{-f(x_i)}\frac{d\Lambda(x_i)}{\Lambda(A)} = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-g(x)})d\Lambda(x)\right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. □

1.1 Superposition of point processes

For simple point processes S_k with intensity measures Λ_k and counting process N_k, the **superposition** of point processes is defined as $S = \cup_k S_k$ a simple point process with the counting process $N = \sum_k N_k$ iff $\sum_k N_k$ is locally finite. If S_k are Poisson then, we know that $\Lambda = \sum_k \Lambda_k$. Next, we will show that the superposition S of Poisson processes is Poisson iff $\sum_k \Lambda_k$ is locally finite.

Theorem 1.3. *The superposition of independent Poisson point processes with intensities Λ_k is a Poisson point process with intensity measure $\sum_k \Lambda_k$ if and only if the latter is a locally finite measure.*

Proof. Consider the superposition $S = \sum_k S_k$ of independent Poisson point processes $S_k \in \mathbb{R}^d$ with intensity measures Λ_k. We will prove only side of this theorem. We assume that $\sum_k \Lambda_k$ is locally finite measure. It is clear that $N(A) = \sum_k N_k(A)$ is finite by locally finite assumption, for all bounded sets $A \in \mathcal{B}$. In particular, we have $dN(x) = \sum_k dN_k(x)$ for all $x \in \mathbb{R}^d$. From the monotone convergence theorem and the independence of counting processes, we have for a non-negative function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{L}_S(f) = \mathbb{E}\exp\left(-\int_{\mathbb{R}^d} f(x)\sum_k dN_k(x)\right) = \prod_k \mathcal{L}_{S_k} = \exp\left(-\int_{\mathbb{R}^d}(1-e^{-f(x)})\sum_k \Lambda_k(x)\right).$$

□

1.2 Thinning of point processes

Consider a probability **retention function** $p : \mathbb{R}^d \to [0,1]$ and a point process S. The **thinning** of point process $S : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$ with the retention function p is a point process $S^p : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$ such that

$$S^p = (S_n \in S : Y(S_n) = 1),$$

where $Y(S_n)$ is an independent indicator random variable at each point S_n and $\mathbb{E}[Y(S_n) \mid S_n] = p(S_n)$.

Theorem 1.4. *The thinning of the Poisson point process of intensity measure Λ with the retention probability function p yields a Poisson point process of intensity measure $\Lambda^{(p)}$ with*

$$\Lambda^{(p)}(A) = \int_A p(x)d\Lambda(x)$$

for all bounded Borel measurable $A \subseteq \mathbb{R}^d$.

Proof. Let $A \subseteq \mathbb{R}^d$ be a bounded Boreal measurable set, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function. Let N^p be the associated counting process to the thinned point process S^p. Hence, for any set $A \in \mathcal{B}$, we have $N^p(A) = \sum_{S_i \in S(A)} Y(S_i)$. That is,

$$dN^p(x) = \sum_{i\in\mathbb{N}} \delta_x \mathbb{1}_{\{x=S_i\}} Y(S_i).$$

Therefore, for any non-negative function $g(x) = f(x)1_{\{x\in A\}}$, we can write

$$\int_{x\in\mathbb{R}^d} g(x)dN^p(x) = \int_{x\in A} f(x)dN^p(x) = \sum_{S_i\in A} f(S_i)Y(S_i).$$

Consider the Laplace functional of the thinned point process S^p for the non-negative function $g(x) = f(x)1_{\{x\in A\}}$, we can write the corresponding Laplace functional as

$$\mathcal{L}_{S^p}(g) = \mathbb{E}\mathbb{E}[\exp\left(-\int_A f(x)dN^p(x)\right) \mid N(A)] = \sum_{n\in\mathbb{Z}_+} P\{N(A)=n\}\prod_{i=1}^n \mathbb{E}[^{-f(S_i)Y(S_i)}\Big| S_i \in A].$$

Here, we denote the points of the point process in subset A as $S(A) = S \cap A$. The first equality follows from the definition of Laplace functional and taking nested expectations. Second equality follows from the fact that the distribution of all points of a Poisson point process are *iid*. Since Y is a Bernoulli process independent of the underlying process S with $\mathbb{E}[Y(S_i)] = p(S_i)$, we get

$$\mathbb{E}[e^{-f(S_i)Y(S_i)}\Big| S_i \in S(A)] = \mathbb{E}[e^{-f(S_i)}p(S_i) + (1-p(S_i))\Big| S_i \in S(A)].$$

From the distribution $\frac{\Lambda'(x)}{\Lambda(A)}$ for $x \in S(A)$ for the Poisson point process S, we get

$$\mathcal{L}_{S^p}(g) = e^{-\Lambda(A)} \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\int_A (p(x)e^{-f(x)} + (1 - p(x))d\Lambda(x) \right)^n = \exp\left(- \int_{\mathbb{R}^d} (1 - e^{-g(x)})p(x)d\Lambda(x) \right).$$

Result follows from taking increasing sequences of sets $A_k \uparrow \mathbb{R}^d$ and monotone convergence theorem. $\square$