Lecture-28: Poisson processes: Equivalences

1 Equivalent characterizations

Definition 1.1. A counting process N has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{B}$,

$$P \bigcap_{i=1}^{k} \{N(A_i) = n_i\} = \prod_{i=1}^{k} P\{N(A_i) = n_i\}.$$

Theorem 1.2. *Distribution of a simple point process is completely determined by void probabilities.*

Theorem 1.3 (Equivalences). *Following are equivalent for a simple counting process $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}}$.*

i_ Process N is Poisson with locally finite intensity measure Λ.

ii_ For each bounded $A \in \mathcal{B}$, we have $P\{N(A) = 0\} = e^{-\Lambda(A)}$.

iii_ For each bounded $A \in \mathcal{B}$, the number of points $N(A)$ is a Poisson with parameter $\Lambda(A)$.

iv_ Process N has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$.

Proof. We will show that $i\_ \implies ii\_ \implies iii\_ \implies iv\_ \implies i\_$.

$i \implies ii\_$ It follows from the definition of Poisson point processes and definition of Poisson random variables.

$ii \implies iii\_$ From Theorem 1.2, we know that void probabilities determine the entire distribution. Further, we observe that

$$\sum_{k \in \mathbb{N}} P\{N(A) = k\} = e^{-\Lambda(A)} \sum_{k \in \mathbb{N}} \frac{\Lambda(A)^k}{k!}.$$

$iii \implies iv\_$ We will show this in two steps.

Mean: Since the distribution of random variable $N(A)$ is Poisson, it has mean $\mathbb{E}N(A) = \Lambda(A)$.

CIP: For disjoint and bounded $A_1, \dots, A_k \in \mathcal{B}$ and $A = \cup_{i=1}^{k} A_i$, we have $N(A) = N(A_1) + \dots N(A_1)$. Taking expectations on both sides, and from the linearity of expectation, we get

$$\Lambda(A) = \Lambda(A_1) + \dots + \Lambda(A_k).$$

From the number of partitions $n_1 + \dots + n_k = n$, we can write

$$P\{N(A) = n\} = \frac{1}{n!} \sum_{n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} P\{N(A_1) = n_1, \dots, N(A_k) = n_k\}.$$

Using the definition of Poisson distribution, we can write the left hand side of the above equation as

$$P\{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!} = \prod_{i=1}^{k} e^{-\Lambda(A_i)} \frac{(\sum_{i=1}^{k} \Lambda(A_i))^n}{n!} = \frac{1}{n!} \sum_{n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^{k} e^{-\Lambda(A_i)} \Lambda(A_i))^{n_i}.$$

Equating each term in the summation, we get

$$P\{N(A_1) = n_1, \dots, N(A_k) = n_k\} = \prod_{i=1}^{k} P\{N(A_i) = n_i\}.$$

$iv \implies i_-$ Due to complete independence property and since void probabilities describe the entire distribution, it suffices to show that $P\{N(A)=0\} = e^{-\Lambda(A)}$ for all bounded $A \in \mathcal{B}$. For disjoint and bounded $A_1, \dots, A_k \in \mathcal{B}$ and $A = \cup_{i=1}^k A_i$, we have

$$\Lambda(A) = \sum_{i=1}^k \Lambda(A_i), \qquad \text{and } -\ln P\{N(A)=0\} = -\sum_{i=1}^k \ln P\{N(A_i)=0\}.$$

This implies that $-\ln P\{N(A)=0\} = \Lambda(A)$, and the result follows.

□

Corollary 1.4 (Poisson process on the half-line). *A random process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ indexed by time $t \in \mathbb{Z}_+$ is a Poisson process with intensity measure Λ iff*

(a) Starting with $N(0) = 0$, the process $N(t)$ takes a non-negative integer value for all $t \in \mathbb{R}_+$;

(b) the increment $N(t+s) - N(t)$ is surely nonnegative for any $s \in \mathbb{R}_+$;

(c) the increments $N(t_1), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$ are independent for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$;

(d) the increment $N(t+s) - N(t)$ is distributed as Poisson random variable with parameter $\Lambda((t, t+s])$.

The Poisson process is homogeneous with intensity λ, iff in addition to conditions $(a),(b),(c)$, the distribution of the increment $N(t+s) - N(t)$ depends on the value $s \in \mathbb{R}_+$ but is independent of $t \in \mathbb{R}_+$. That, is the increments are stationary.

Proof. We have already seen that definition of Poisson processes implies all four conditions. Conditions (a) and (b) imply that N is a simple counting process on the half-line, condition (c) is the complete independence property of the point process, and condition (d) provides the intensity measure. The result follows from the equivalence iv_- in Theorem 1.3. □

A Memoryless distribution

Definition A.1. A random variable X with continuous support on $\mathbb{R}_+$, is called **memoryless** if for all positive reals $t, s \in \mathbb{R}_+$, we have

$$P\{X > s\} = P(\{X > t+s\} \mid \{X > t\}).$$

Proposition A.2. *The unique memoryless distribution function with continuous support on $\mathbb{R}_+$ is the exponential distribution.*

Proof. Let X be a random variable with a memoryless distribution function $F : \mathbb{R}_+ \to [0,1]$. It follows that $\bar{F}(t) \triangleq 1 - F(t)$ satisfies the semi-group property

$$\bar{F}(t+s) = \bar{F}(t)\bar{F}(s).$$

Since $\bar{F}(x) = P\{X > x\}$ is non-increasing in $x \in \mathbb{R}_+$, we have $\bar{F}(x) = e^{\theta x}$, for some $\theta < 0$ from Lemma A.3. □

Lemma A.3. *A unique non-negative right continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the semigroup property*

$$f(t+s) = f(t)f(s), \text{ for all } t, s \in \mathbb{R}_+$$

is $f(t) = e^{\theta t}$, where $\theta = \log f(1)$.

Proof. Clearly, we have $f(0) = f^2(0)$. Since f is non-negative, it means $f(0) = 1$. By definition of θ and induction for $m, n \in \mathbb{Z}^+$, we see that

$$f(m) = f(1)^m = e^{\theta m}, \qquad e^{\theta} = f(1) = f(1/n)^n.$$

Let $q \in \mathbb{Q}$, then it can be written as $m/n, n \neq 0$ for some $m, n \in \mathbb{Z}^+$. Hence, it is clear that for all $q \in \mathbb{Q}^+$, we have $f(q) = e^{\theta q}$. either unity or zero. Note, that f is a right continuous function and is non-negative.

Now, we can show that f is exponential for any real positive t by taking a sequence of rational numbers $(q_n : n \in \mathbb{N})$ decreasing to t. From right continuity of f, we obtain

$$f(t) = \lim_{q_n \downarrow t} f(q_n) = \lim_{q_n \downarrow t} e^{\theta q_n} = e^{\theta t}.$$

□