Lecture-04: Random Variable

1 Random Variable

Definition 1.1 (Random variable). Consider a probability space $(\Omega, \mathcal{F}, P)$. A **random variable** $X : \Omega \to \mathbb{R}$ is a real-valued function from the sample space to real numbers, such that for each $x \in \mathbb{R}$ the event

$$A_x \triangleq \{\omega \in \Omega : X(\omega) \leqslant x\} = X^{-1}(-\infty, x] = X^{-1}(B_x) \in \mathcal{F}.$$

We say that the random variable X is $\mathcal{F}$-measurable and the probability of this event is denoted by

$$F_X(x) \triangleq P(A_x) = P(\{X \leqslant x\}) = P \circ X^{-1}(-\infty, x] = P \circ X^{-1}(B_x).$$

The function $F_X : \mathbb{R} \to [0,1]$ is called the **distribution function** (CDF) of a random variable X.

Remark 1. Since any outcome $\omega \in \Omega$ is random, so is the real value $X(\omega)$.

Remark 2. Probability is defined only for events and not for random variables. The events of interest for random variables are the upper-level sets $A_x = X^{-1}(B_x)$ for any real x.

Definition 1.2 (Borel sets). The smallest σ-algebra generated by the half-open sets $B_x = (-\infty, x]$ for all $x \in \mathbb{R}$ is called a **Borel σ-algebra** and denoted by $\mathcal{B}(\mathbb{R}) = \sigma(\{B_x : x \in \mathbb{R}\})$. The elements of the Borel σ-algebra are called **Borel sets**.

Definition 1.3. The event space generated by a random variable X defined on probability space $(\Omega, \mathcal{F}, P)$, is denoted by $\sigma(X) \triangleq \sigma(\{A_x : x \in \mathbb{R}\})$.

Remark 3. The event space generated by a random variable is collection of the inverse of Borel sets, i.e. $\sigma(X) = \left\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\right\}$. This follows from the fact that $A_x = X^{-1}(B_x)$ and the inverse map respects countable set operations such as unions, complements, and intersections. That is, if $B \in \mathcal{B}(\mathbb{R}) = \sigma(\{B_x : x \in \mathbb{R}\})$, then $X^{-1}(B) \in \sigma(\{A_x : x \in \mathbb{R}\})$. Similarly, if $A \in \sigma(X) = \sigma(\{A_x : x \in \mathbb{R}\})$, then $A = X^{-1}(B)$ for some $B \in \sigma(\{B_x : x \in \mathbb{R}\})$.

Example 1.4 (Constant random variable). Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $X(\omega) = c$ for all outcomes $\omega \in \Omega$. The distribution function is a right-continuous step function at c with step-value unity. That is,

$$F_X(x) = \mathbb{1}_{\{x \geqslant c\}}.$$

We observe that $P(\{X = c\}) = 1$. Does this make sense? Is $\{\omega \in \Omega : X(\omega) = x\}$ an event for a general random variable?

Consider a monotonically increasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = x$. Then, the sequence of events $(A_{x_n} \in \mathcal{F} : n \in \mathbb{N})$ is monotonically increasing and hence the $\lim_n A_{x_n} = \{X < x\} \in \mathcal{F}$ is an event. It follows that $\{X = x\} = A_x \cap \{X < x\}^c \in \mathcal{F}$ is also an event.

Definition 1.5 (Indicator functions). For a probability space $(\Omega, \mathcal{F}, P)$ and any event $A \in \mathcal{F}$, we can define an **indicator function** $\mathbb{1}_A : \Omega \to \{0,1\}$ as

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Example 1.6. We will show that indicator function $\mathbb{1}_A$ is a random variable for any event $A \in \mathcal{F}$. Let $x \in \mathbb{R}$, and $B_x = (-\infty, x]$, then it follows that

$$\mathbb{1}_A^{-1}(B_x) = \begin{cases} \Omega, & x \geqslant 1, \\ A^c, & x \in [0,1), \\ \emptyset, & x < 0. \end{cases}$$

That is, $\mathbb{1}_A^{-1}(B_x) \in \mathcal{F}$ for all $x \in \mathbb{R}$, and hence it is a random variable. Further, we can write $\sigma(\mathbb{1}_A) = \{\emptyset, A, A^c, \Omega\}$. In addition, we have

$$F_X(x) = \begin{cases} 1, & x \geqslant 1, \\ 1 - P(A), & x \in [0,1), \\ 0, & x < 0. \end{cases}$$

1.1 Event space generated by random variables

Definition 1.7 (Event space generated by a random variable). Let $X : \Omega \to \mathbb{R}$ be a random variable on the probability space $(\Omega, \mathcal{F}, P)$, then the smallest event space generated by the events of the form $X^{-1}(-\infty, x]$ for $x \in \mathbb{R}$ is called the **event space generated** by this random variable X, and denoted by $\sigma(X)$.

Example 1.8 (Indicator function). Let $A \in \mathcal{F}$ be an event, then $X = \mathbb{1}_A$ is a random variable and

$$X^{-1}(-\infty, x] = \begin{cases} \Omega, & x \geqslant 1, \\ A^c, & x \in [0,1), \\ \emptyset, & x < 0. \end{cases}$$

This implies that the smallest event space generated by this random variable is $\sigma(X) = \{\emptyset, A, A^c, \Omega\}$.

Example 1.9 (Simple random variables). Let X be a simple random variable, then $X = \sum_{x \in \mathcal{X}} x \mathbb{1}_{A_x}$ where $(A_x = X^{-1}\{x\} \in \mathcal{F} : x \in \mathcal{X})$ is a finite partition of the sample space Ω. Without loss of generality, we can denote $\mathcal{X} = \{x_1, \dots, x_n\}$ where $x_1 \leqslant \dots \leqslant x_n$. Then,

$$X^{-1}(-\infty, x] = \begin{cases} \Omega, & x \geqslant x_n, \\ \cup_{j=1}^{i} A_{x_i}, & x \in [x_i, x_{i+1}), i \in [n-1], \\ \emptyset, & x < x_1. \end{cases}$$

Then the smallest event space generated by the simple random variable X is $\{\cup_{x \in S} A_x : S \subseteq \mathcal{X}\}$.

1.2 Discrete random variables

Definition 1.10 (Discrete random variables). If a random variable $X : \Omega \to \mathcal{X} \subseteq \mathbb{R}$ takes countable values on real-line, then it is called a **discrete random variable**. That is, the range of random variable $\mathcal{X}$ is countable, and the random variable is completely specified by the **probability mass function**

$$P_X(x) = P(\{X = x\}), \text{ for all } x \in \mathcal{X}.$$

Example 1.11 (Bernoulli random variable). For the probability space $(\Omega,\mathcal{F},P)$, the **Bernoulli random variable** is a mapping $X:\Omega\to\{0,1\}$ and $P_X(1)=p$. We observe that Bernoulli random variable is an indicator for the event $A\triangleq X^{-1}\{1\}$, and $P(A)=p$. Therefore, the distribution function F_X is given by

$$F_X=(1-p)\mathbb{1}_{[0,1)}+\mathbb{1}_{[1,\infty)}.$$

Definition 1.12 (Simple functions). For a probability space $(\Omega,\mathcal{F},P)$, a finite n, events $E_1,\dots,E_n\in\mathcal{F}$, and real constants $x_1,\dots,x_n$ we can define a simple function $X:\Omega\to\{x_1,\dots,x_n\}$ to be

$$X(\omega)=\sum_{i=1}^{n}x_i\mathbb{1}_{E_i}(\omega).$$

Lemma 1.13. *Any discrete random variable is a linear combination of indicator function over a partition of the sample space.*

Proof. For a discrete random variable $X:\Omega\to\mathcal{X}\subset\mathbb{R}$ on a probability space $(\Omega,\mathcal{F},P)$, the range $\mathcal{X}$ is countable, and we can define events $E_x\triangleq\{\omega\in\Omega:X(\omega)=x\}\in\mathcal{F}$ for each $x\in\mathcal{X}$. Then the mutually disjoint sequence of events $(E_x\subseteq\Omega:x\in\mathcal{X})$ partitions the sample space Ω. We can write

$$X(\omega)=\sum_{x\in\mathcal{X}}x\mathbb{1}_{E_x}(\omega).$$

□

1.3 Continuous random variables

Definition 1.14. For a continuous random variable X, there exists **density function** $f_X:\mathbb{R}\to[0,\infty)$ such that

$$F_X(x)=\int_{-\infty}^{x}f_X(u)du.$$

Example 1.15 (Gaussian random variable). For a probability space $(\Omega,\mathcal{F},P)$, **Gaussian random variable** is a continuous random variable $X:\Omega\to\mathbb{R}$ defined by its density function

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R}.$$

2 Properties of distribution function

Lemma 2.1 (Properties of distribution function). *The distribution function F_X for any random variable X satisfies the following properties.*

1. *The distribution function is monotonically non-decreasing in $x\in\mathbb{R}$.*
2. *The distribution function is right-continuous at all points $x\in\mathbb{R}$.*
3. *The upper limit is $\lim_{x\to\infty}F_X(x)=1$ and the lower limit is $\lim_{x\to-\infty}F_X(x)=0$.*

Proof. Let X be a random variable defined on the probability space $(\Omega,\mathcal{F},P)$.

1. Let $x_1,x_2\in\mathbb{R}$ such that $x_1\leqslant x_2$. Then for any $\omega\in A_{x_1}$, we have $X(\omega)\leqslant x_1\leqslant x_2$, and it follows that $\omega\in A_{x_2}$. This implies that $A_{x_1}\subseteq A_{x_2}$. The result follows from the monotonicity of the probability.

2. For any $x \in \mathbb{R}$, consider any monotonically decreasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = x$. It follows that the sequence of events $\left(A_{x_n} = X^{-1}(-\infty, x_n] \in \mathcal{F} : n \in \mathbb{N}\right)$, is monotonically decreasing and hence $\lim_{n\in\mathbb{N}} A_{x_n} = \cap_{n\in\mathbb{N}} A_{x_n} = A_x$. The right-continuity then follows from the continuity of probability, since
$$F_X(x) = P(A_x) = P(\lim_{n\in\mathbb{N}} A_{x_n}) = \lim_{n\in\mathbb{N}} P(A_{x_n}) = \lim_{x_n \downarrow x} F(x_n).$$

3. Consider a monotonically increasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = \infty$, then $(A_{x_n} \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically increasing sequence of sets and $\lim_n A_{x_n} = \cup_{n\in\mathbb{N}} A_{x_n} = \Omega$. From the continuity of probability, it follows that
$$\lim_{x_n \to \infty} F_X(x_n) = \lim_n P(A_{x_n}) = P(\lim_n A_{x_n}) = P(\Omega) = 1.$$
Similarly, we can take a monotonically decreasing sequence $(x_n \in \mathbb{R} : n \in \mathbb{N})$ such that $\lim_n x_n = -\infty$, then $(A_{x_n} \in \mathcal{F} : n \in \mathbb{N})$ is a monotonically decreasing sequence of sets and $\lim_n A_{x_n} = \cap_{n\in\mathbb{N}} A_{x_n} = \emptyset$. From the continuity of probability, it follows that $\lim_{x_n \to -\infty} F_X(x_n) = 0$.

□

Remark 4. If two reals $x_1 < x_2$ then $F_X(x_1) \leqslant F_X(x_2)$ with equality if and only if $P\{(x_1 < X \leqslant x_2\}) = 0$. This follows from the fact that $A_{x_2} = A_{x_1} \cup X^{-1}(x_1, x_2]$.