Lecture-16: Random Processes

1 Introduction

Recall that the projection operator $\pi_t : \mathbb{R}^T \to \mathbb{R}$ maps any T-vector $x \in \mathbb{R}^T$ to $\pi_t(x) = x_t$.

Definition 1.1 (Random process). Let $(\Omega, \mathcal{F}, P)$ be a probability space. For an arbitrary index set T and state space $\mathcal{X} \subseteq \mathbb{R}$, map $X : \Omega \to \mathcal{X}^T$ is called a **random process** if the projections $X_t : \Omega \to \mathcal{X}$ defined by $\omega \mapsto X_t(\omega) \triangleq (\pi_t \circ X)(\omega)$ are random variables on the given probability space.

Definition 1.2. For each outcome $\omega \in \Omega$, we have a function $X(\omega) : T \mapsto \mathcal{X}$ called the **sample path** or the **sample function** of the process X.

Remark 1. A random process X defined on probability space $(\Omega, \mathcal{F}, P)$ with index set T and state space $\mathcal{X} \subseteq \mathbb{R}$ can be thought of as

(a) a map $X : \Omega \times T \to \mathcal{X}$,

(b) a map $X : T \to \mathcal{X}^\Omega$, i.e. a collection of random variables $X_t : \Omega \to \mathcal{X}$ for each time $t \in T$,

(c) a map $X : \Omega \to \mathcal{X}^T$, i.e. a collection of sample functions $X(\omega) : T \to \mathcal{X}$ for each random outcome $\omega \in \Omega$.

1.1 Classification

State space $\mathcal{X}$ can be countable or uncountable, corresponding to discrete or continuous valued process. If the index set $T \subseteq \mathbb{R}$ is countable, the stochastic process is called **discrete**-time stochastic process or random sequence. When the index set T is uncountable, it is called **continuous**-time stochastic process. The index set T doesn't have to be time, if the index set is space, and then the stochastic process is spatial process. When $T = \mathbb{R}^n \times [0, \infty)$, stochastic process X is a spatio-temporal process.

Example 1.3. We list some examples of each such stochastic process.

i. Discrete random sequence: brand switching, discrete time queues, number of people at bank each day.

ii. Continuous random sequence: stock prices, currency exchange rates, waiting time in queue of nth arrival, workload at arrivals in time sharing computer systems.

iii. Discrete random process: counting processes, population sampled at birth-death instants, number of people in queues.

iv. Continuous random process: water level in a dam, waiting time till service in a queue, location of a mobile node in a network.

1.2 Measurability

For random process $X : \Omega \to \mathcal{X}^T$, the projections $X_t \triangleq \pi_t \circ X$ are random variables. Therefore, the set of outcomes $A_t(x) \triangleq X_t^{-1}(-\infty, x] \in \mathcal{F}$ for all $t \in T$ and $x \in \mathbb{R}$.

Definition 1.4. A random map $X : \Omega \to \mathcal{X}^T$ is called $\mathcal{F}$-**measurable** and hence a random process, if the set of outcomes $A_t(x) = X_t^{-1}(-\infty, x] \in \mathcal{F}$ for all $t \in T$ and $x \in \mathbb{R}$.

Definition 1.5. The **event space generated by a random process** $X : \Omega \to \mathcal{X}^T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is given by

$$\sigma(X) \triangleq \sigma(A_t(x) : t \in T, x \in \mathbb{R}).$$

Remark 2. Recall that $\pi_t^{-1}(-\infty, x] = \bigtimes_{s \in T}(-\infty, x_s]$ where $x_s = x$ for $s = t$ and $x_s = \infty$ for all $s \neq t$. The $\mathcal{F}$-measurability of process X implies that for any countable set $S \subseteq T$, we have $A_S(x_S) \triangleq \cap_{s \in S} A_s(x_s) \in \mathcal{F}$ for $x_S \in \mathcal{X}^S$. We can construct a $x = \pi_S^{-1}(x) \in \mathbb{R}^T$ such that $\pi_s(x) = x_s$ for $s \in S$ and $\pi_t(x) = \infty$ for $t \notin S$, and define

$$A(x) \triangleq \cap_{t \in T} A_t(x_t) = \cap_{s \in S} A_s(x_s) = A_S(x_S) \in \mathcal{F}.$$

Recall that $A(x)$ is an event only when $\{t \in T : \pi_t(x) < \infty\}$ is a countable set.

Example 1.6 (Bernoulli sequence). Let index set be $\mathbb{N}$ and the sample space be the collection of infinite bi-variate sequences of successes (S) and failures (F) defined by $\Omega = \{S, F\}^{\mathbb{N}}$. An outcome $\omega \in \Omega$ is an infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ such that $\omega_n = \pi_n(\omega) \in \{S, F\}$ for each $n \in \mathbb{N}$. We define the random process $X : \Omega \to \{0, 1\}^{\mathbb{N}}$ such that $X_n(\omega) = \mathbb{1}_{\{S\}}(\omega_n) = \mathbb{1}_{\{\omega_n = S\}}$. Hence, we can write the process X as the collection of random variables $X : \mathbb{N} \to \{0, 1\}^{\Omega}$ or the collection of sample paths $X : \Omega \to \{0, 1\}^{\mathbb{N}}$.

Since each $X_n : \Omega \to \{0, 1\}$ is a bi-variate random variable on the probability space $(\Omega, \mathcal{F}, P)$, the event space is generated by events $E_n \triangleq \{\omega \in \Omega : X_n(\omega) = 1\} = \left\{\omega \in \{S, F\}^{\mathbb{N}} : \omega_n = S\right\} \in \mathcal{F}$. That is,

$$\sigma(X) = \sigma(E_n : n \in \mathbb{N}).$$

Definition 1.7. For a random process $X : \Omega \to \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we define a **finite dimensional distribution** $F_{X_S} : \mathbb{R}^S \to [0, 1]$ for a finite $S \subseteq T$ by

$$F_{X_S}(x_S) \triangleq P(A_S(x_S)), \quad x_S \in \mathbb{R}^S.$$

1.3 Independence

Recall the following definitions given the probability space $(\Omega, \mathcal{F}, P)$. Events $(A_i \in \mathcal{F} : i \in [n]) \in \mathcal{F}$ are independent if $P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$. A collection of events $(A_t \in \mathcal{F} : t \in T)$ is independent if $(A_s : s \in S)$ are independent for all finite $S \subseteq T$.

Definition 1.8. Event spaces $(\mathcal{G}_i \subseteq \mathcal{F} : i \in [n])$ are independent if any collection of events $G \in \bigtimes_{i \in [n]} \mathcal{G}_i$ are independent. That is,

$$P(\cap_{i=1}^n G_i) = \prod_{i=1}^n P(G_i), \text{ for all } G_i \in \mathcal{G}_i, i \in [n].$$

Definition 1.9. Random vector $X : \Omega \to \mathbb{R}^n$ is independent if event spaces $\sigma(X_1), \dots, \sigma(X_n)$ are independent. That is, for all $x \in \mathbb{R}^n$, we have

$$F_X(x) = P(\cap_{i=1}^n X_i^{-1}(-\infty, x_i]) = \prod_{i=1}^n P \circ X_i^{-1}(-\infty, x_i] = \prod_{i=1}^n F_{X_i}(x_i).$$

That is, independence of random vector is equivalent to factorization of joint distribution function into product of individual marginal distribution functions.

Definition 1.10. A collection of event spaces $(\mathcal{G}_t \subseteq \mathcal{F} : t \in T)$ is independent if any finite subcollection $(\mathcal{G}_t : t \in S)$ for finite $S \subseteq T$ is independent.

Definition 1.11. A random process is **independent** if the collection of event spaces $(\sigma(X_t) : t \in T)$ is independent. That is, for all $x_S \in \mathbb{R}^S$, we have

$$F_{X_S}(x_S) = P(\cap_{s\in S}\{X_s \leqslant x_s\}) = \prod_{s\in S} P\{X_s \leqslant x_s\} = \prod_{s\in S} F_{X_s}(x_s).$$

That is, independence of a random process is equivalent to factorization of any finite dimensional distribution function into product of individual marginal distribution functions.

Definition 1.12. Two stochastic processes $X : \Omega \to \mathcal{X}^{T_1}, Y : \Omega \to \mathcal{Y}^{T_2}$ are **independent**, if the corresponding event spaces $\sigma(X), \sigma(Y)$ are independent. That is, for any $x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}$ for finite $S_1 \subseteq T_1, S_2 \subseteq T_2$, the events $A_{S_1}(x) \triangleq \cap_{s\in S_1} X_s^{-1}(-\infty, x_s]$ and $B_{S_2}(y) \triangleq \cap_{s\in S_2} Y_s^{-1}(-\infty, y_s]$ are independent. That is, the joint finite dimensional distribution of X and Y factorizes, and

$$P(A_{S_1}(x) \cap B_{S_2}(y)) = P(A_{S_1}(x))P(B_{S_2}(y)) = F_{X_{S_1}}(x)F_{Y_{S_2}}(y), \quad x \in \mathbb{R}^{S_1}, y \in \mathbb{R}^{S_2}.$$

1.4 Distribution

To define a measure on a random process, we can either put a measure on sample paths, or equip the collection of random variables with a joint measure. We are interested in identifying the joint distribution $F : \mathbb{R}^T \to [0,1]$. To this end, for any $x \in \mathbb{R}^T$ we need to know

$$F_X(x) \triangleq P\left(\bigcap_{t\in T}\{\omega \in \Omega : X_t(\omega) \leqslant x_t\}\right) = P(\bigcap_{t\in T} X_t^{-1}(-\infty, x_t]) = P(A(x)).$$

First of all, we don't know whether $A(x)$ is an event since $A(x) \in \mathcal{F}$ when $x \in \mathbb{R}^T$ such that $\{t \in T : x_t < \infty\}$ is countable. Second, even for a simple independent process with countably infinite T, any function of the above form would be zero if x_t is finite for all $t \in T$. Therefore, we only look at the values of $F_X(x)$ for $x \in \mathbb{R}^T$ where $\{t \in T : x_t < \infty\}$ is finite. That is, for any finite set $S \subseteq T$, we focus on the events $A_S(x_S)$ and their probabilities. However, these are precisely the finite dimensional distributions. Set of all finite dimensional distributions of the stochastic process $X : \Omega \to \mathcal{X}^T$ characterizes its distribution completely.

1.5 Joint moments

Simpler characterizations of a stochastic process X are in terms of its moments. That is, the first moment such as mean, and the second moment such as correlations and covariance functions.

$$m_X(t) \triangleq \mathbb{E}X_t, \qquad R_X(t,s) \triangleq \mathbb{E}X_tX_s, \qquad C_X(t,s) \triangleq \mathbb{E}(X_t - m_X(t))(X_s - m_X(s)).$$

Example 1.13 (Bernoulli sequence). For the Bernoulli sequence X defined in Example 1.6 on probability space $(\Omega, \mathcal{F}, P)$, we take the probability of generating events be

$$P(\cap_{i\in S} E_i) = p^{|S|}, \quad S \subseteq \mathbb{N}.$$

This probability of events implies that finite dimensional distributions factorize and they are identical. In particular, we have

$$P(\cap_{s\in S}\{X_s = x_s\}) = p^{\sum_{s\in S} x_s}(1-p)^{|S| - \sum_{s\in S} x_s}, \quad S \subseteq \mathbb{N}.$$

Hence, the random sequence X is independent and identically distributed with $P\{X_n = 1\} = p \in (0,1)$. For any sequence $x \in \{0,1\}^{\mathbb{N}}$, we have $P\{X = x\} = 0$. Let $q \triangleq (1-p)$, then the probability of observing m heads and r tails is given by $p^m q^r$. We can easily compute the mean, the auto-correlation, and the auto-covariance functions for the independent Bernoulli process defined in Example 1.13 as

$$m_X(n) = \mathbb{E}X_n = p, \qquad R_X(m,n) = \mathbb{E}X_mX_n = \mathbb{E}X_m\mathbb{E}X_n = p^2, \qquad C_x(m,n) = 0.$$