Lecture-17: Stopping Times

1 Random Walk

Definition 1.1. Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an *i.i.d.* random sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ and the state space $\mathcal{X} = \mathbb{R}^d$. A random sequence $S : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$ is called a **random walk** with step-size sequence X, if $S_0 \triangleq 0$ and $S_n \triangleq \sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$.

Remark 1. We can think of S_n as the random location of a particle after n steps, where the particle starts from origin and takes steps of size X_i at the ith step. From the *i.i.d.* nature of step-size sequence, we observe that $\mathbb{E}S_n = n\mathbb{E}X_1$ and $C_S(n,m) = (n \wedge m)\,\mathrm{Var}[X_1]$.

Remark 2. For the process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ it suffices to look at finite dimensional distributions for finite sets $[n] \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$. If the *i.i.d.* step-size sequence X has a common density function, then from the transformation of random vectors, we can find the finite dimensional density

$$f_{S_1,\dots,S_n}(s_1, s_2, \dots, s_n) = f_{X_1,\dots,X_n}(s_1, s_2 - s_1, \dots, s_n - s_{n-1}) \det[J(s)] = f_{X_1}(s_1) \prod_{i=2}^{n} f_{X_1}(s_i - s_{i-1}).$$

Remark 3. If X_n denotes the success of nth experiment, then S_n denotes the number of successes in first n trials. In particular, $S_n \in \{0, \dots, n\}$, i.e. the set of all outcomes is index dependent. Further, $S_n \geqslant 0$ for all n and is a non-decreasing process, since $S_n = S_{n-1} + X_n$. In particular, it is a discrete counting process.

Example 1.2. Example of discrete counting processes.

i_ For products manufactured in an assembly line, S_n indicates the number of defective products in the first n manufactured.

ii_ At a fork on the road, S_n indicates the number of vehicles that turned left for first n vehicles that arrived at the fork.

Remark 4. For a one-dimensional random walk $S : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ with *i.i.d.* step size sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ such that $P\{X_1 = 1\} = p$, the distribution for the random walk at nth step S_n is Binomial (n, p). That is, $P\{S_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \{0, \dots, n\}.$

2 Stopping Times

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, then a collection of event spaces denoted $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ is called a filtration on this probability space for an ordered index set T, if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leqslant t$.

Definition 2.2. For a random process $X : \Omega \to \mathcal{X}^T$ defined on probability space $(\Omega, \mathcal{F}, P)$ with state spapce $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set T, we can find the event space generated by all random variables until time t as $\sigma(X_s, s \leqslant t)$. The natural filtration associated with the random process X is given by $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$.

Example 2.3. For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the natural filtration is a sequence $\mathcal{F}_\bullet = (\mathcal{F}_n : n \in \mathbb{N})$ of event spaces $\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n)$ for all $n \in \mathbb{N}$. For the random walk S with step size sequence X, the natural filtration is identical to that of the step size sequence. That is,

$\sigma(S_1,\dots,S_n) = \sigma(X_1,\dots,X_n)$. This follows from the fact that there is a bijection between $(X_1,\dots,X_n)$ and $(S_1,\dots,S_n)$, where $S_j = \sum_{i=1}^{j} X_i, j \in [n]$ and $X_j = S_j - S_{j-1}, j \in [n]$.

If the random sequence X is independent, then the random sequence $(X_{n+j} : j \in \mathbb{N})$ is independent of the event space $\sigma(X_1,\dots,X_n)$.

Definition 2.4. For an ordered index set T, a random variable $\tau : \Omega \to T$ is called a **stopping time** with respect to a filtration $\mathcal{F}_\bullet$ if

(a) the event $\tau^{-1}(-\infty, t] \in \mathcal{F}_t$ for all $t \in T$, and

(b) the random variable τ is finite almost surely, i.e. $P\{\tau < \infty\} = 1$.

Consider the natural filtration $\mathcal{F}_\bullet$ for a random process $X : \Omega \to \mathcal{X}^T$, defined by $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$ for all $t \in T$. We can consider the ordered index set T as a time sequence. Intuitively, if we observe the process X sequentially, then the event $\{\tau \leqslant t\}$ can be completely determined by the observation $(X_s, s \leqslant t)$ until time t. The intuition behind a stopping time is that it's realization is determined by the past and present events but not by future events. That is, given the history of the process until time t, we can tell whether the stopping time is t or not. In particular, $\mathbb{E}[\mathbb{1}_{\{\tau \leqslant t\}} \mid \mathcal{F}_t]$ is either one or zero.

Example 2.5. while traveling on the bus, the random variable measuring "time until bus crosses next stop after Majestic" is a stopping time as it's value is determined by events before it happens. On the other hand "time until bus crosses the stop before Majestic" would not be a stopping time in the same context. This is because we have to cross this stop, reach Majestic and then realize we have crossed that point.

Theorem 2.6. *For a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$, a discrete random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is a **stopping time** with respect to this random sequence X iff*

(i) the event $\{\tau = n\} \in \sigma(X_1,\dots,X_n)$ for all $n \in \mathbb{N}$, and

(ii) the stopping time is finite almost surely, i.e. $P\{\tau < \infty\} = 1$.

Proof. From Definition 2.4, we have $\{\tau = n\} = \{\tau \leqslant n\} \setminus \{\tau \leqslant n-1\} \in \mathcal{F}_n$. Conversely, from the theorem hypothesis, it follows that $\{\tau \leqslant n\} = \cup_{m=1}^{n} \{\tau = m\} \in \mathcal{F}_n$. □

Example 2.7. Consider a random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the natural filtration $\mathcal{F}_\bullet$, and a measurable set $A \in \mathcal{B}(\mathcal{X})$. The first hitting time $\tau_X^A : \Omega \to \mathbb{N} \cup \{\infty\}$ for the sequence X to hit set A is defined by

$$\tau_X^A \triangleq \inf\{n \in \mathbb{N} : X_n \in A\}.$$

If τ_X^A is almost surely finite, then τ_X^A is a stopping time. This follows from the fact that $\{\tau_X^A = n\} = \cap_{k=1}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n$.

2.1 Properties of stopping time

Lemma 2.8. *Let τ_1, τ_2 be two stopping times with respect to filtration $(\mathcal{F}_t : t \in T)$. Then the following hold true.*

*i_* $\min\{\tau_1, \tau_2\}$ *is a stopping time.*

ii_ If T is separable, then $\tau_1 + \tau_2$ is a stopping time.

Proof. Let $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration, and τ_1, τ_2 associated stopping times.

i_ Result follows since the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$.

ii. It suffices to show that the event $\{\tau_1 + \tau_2 \leqslant t\} \in \mathcal{F}_t$ for $T = \mathbb{R}_+$. To this end, we observe that

$$\{\tau_1 + \tau_2 \leqslant t\} = \bigcup_{s \in \mathbb{Q}_+ :\, s \leqslant t} \{\tau_1 \leqslant t - s, \tau_2 \leqslant s\} \in \mathcal{F}_t.$$

□

Example 2.9. Let $S : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ denote the random walk associated with the *i.i.d.* Bernoulli step size sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ where $\mathbb{E}X_1 = p$. For the set $A = \{1\}$, we have $\tau_X^A = \tau_S^A = \inf\{n \in \mathbb{N} : S_n = 1\}$. Further, we have $P\{\tau_X^A = n\} = (1-p)^{n-1}p$, and therefore $P\{\tau_X^A < \infty\} = \sum_{n \in \mathbb{N}} P\{\tau_X^A = n\} = 1$. That is, $\tau_X^{\{1\}}$ is a stopping time.

Lemma 2.10 (Wald's Lemma). *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{Z}_+}$ with* i.i.d. *step-sizes $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ having finite $\mathbb{E}|X_1|$. Let τ be a finite mean stopping time with respect to this random walk. Then,*

$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\,\mathbb{E}[\tau].$$

Example 2.11 (Incorrect Proof). At first glance, this looks like an easy statement to prove since X is an *i.i.d.* sequence. Using dominated convergence theorem and almost sure finiteness of stopping time τ, we can write

$$\mathbb{E}S_\tau = \mathbb{E}[\sum_{n \in \mathbb{N}} S_n \mathbb{1}_{\{\tau = n\}}] = \sum_{n \in \mathbb{N}} \mathbb{E}[S_n \mathbb{1}_{\{\tau = n\}}].$$

If τ was a random time independent of $\sigma(S_n)$, then we can write from monotone convergence theorem

$$\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E} \sum_{n \in \mathbb{N}} n \mathbb{1}_{\{\tau = n\}} = \mathbb{E}X_1 \mathbb{E}\tau.$$

However, we can't proceed any further when τ is stopping time, since S_n and $\{\tau = n\}$ are not independent events.

Example 2.12 (When τ is not a stopping time). Consider a random walk S associated with an *i.i.d.* step size sequence X where $\mathbb{E}X_1 = p$, and the associated natural filtration $\mathcal{F}_\bullet$. we define a random time $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$, such that

$$\tau \triangleq \inf\{n \in \mathbb{N} : S_{n+1} = 1\}.$$

We first observe that τ is not a stopping time, since the event $\{\tau = n\} = \{S_1 = \cdots = S_n = 0, S_{n+1} = 1\} \in \mathcal{F}_{n+1}$ and this event doesn't belong to $\mathcal{F}_n$. Second, we observe that $S_\tau = 0, \tau \geqslant 1, \mathbb{E}X_1 = p$ and hence $\mathbb{E}S_\tau \neq \mathbb{E}\tau\mathbb{E}X_1$.

Proof. From the independence of step sizes, it follows that X_n is independent of $\sigma(X_0, X_1, \dots, X_{n-1})$. Since τ is a stopping time with respect to random walk S, we observe that $\{\tau \geqslant n\} = \{\tau > n-1\} \in \sigma(X_0, X_1, \dots, X_{n-1})$, and hence it follows that random variable X_n and $\mathbb{1}_{\{\tau \geqslant n\}}$ are independent and $\mathbb{E}[X_n \mathbb{1}_{\{\tau \geqslant n\}}] = \mathbb{E}X_n \mathbb{E}\mathbb{1}_{\{\tau \geqslant n\}}$. Therefore,

$$\mathbb{E}\sum_{n=1}^{\tau} X_n = \mathbb{E}\sum_{n \in \mathbb{N}} X_n \mathbb{1}_{\{\tau \geqslant n\}} = \sum_{n \in \mathbb{N}} \mathbb{E}X_n \mathbb{E}\left[\mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau \geqslant n\}}\right] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it by the application of dominated convergence theorem. □

Example 2.13. For an integer random walk $S : \Omega \to \mathbb{Z}^{\mathbb{N}}$ with *i.i.d.* steps $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$, consider the hitting time $\tau_S^{\{i\}}$ by random walk S to set $A = \{i\}$. The mean of the stopping time $\tau_S^{\{k\}} \triangleq \min\{n \in \mathbb{N} : S_n = k\}$ is given by $\mathbb{E}\tau_S^{\{k\}} = k/\mathbb{E}X_1$. This follows from the Wald's Lemma and the fact that $S_{\tau^i} = i$.

Theorem 2.14 (Strong Independence Property). *Let $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ be an independent random sequence with natural filtration $\mathcal{F}_\bullet$, and $\tau : \Omega \to \mathbb{N}$ a stopping time adapted to the natural filtration of process X. Then, the random collection of random variables $(X_{\tau+n} : n \in \mathbb{N})$ is independent of the random past $(X_n : n \leqslant \tau)$.*

Example 2.15 (When τ is not a stopping time). Consider a random walk S associated with an *i.i.d.* step size sequence X where $\mathbb{E}X_1 = p$, and the associated natural filtration $\mathcal{F}_\bullet$. we define a random time $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$, such that $\tau \triangleq \inf\{n \in \mathbb{N} : S_{n+1} = 1\}$. Recall that τ is not a stopping time. Further, we observe that $S_1 = \cdots = S_\tau = 0$, and $S_{\tau+1} = 1$. In particular, $S_{\tau+1} = 1 - S_\tau$ and hence $\sigma(S_1, \ldots, S_\tau)$ and $\sigma(S_{\tau+1})$ are not independent.

Proof. Since τ is an almost surely finite discrete random variable that takes values in $\mathbb{N}$, we have exception set $E = \{\tau = \infty\}$ and the complement $E^c = \cup_{n\in\mathbb{N}} \{\tau = n\}$. For each $n \in \mathbb{N}$, the event $\{\tau = n\} \in \mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, and the collection $(X_{n+j} : j \in \mathbb{N})$ is independent of $(X_j : j \in [n])$. Consider an event $F \in \sigma(X_1, \ldots, X_\tau)$ and another event $G_\tau \in \sigma(X_{\tau+1}, \ldots)$. Then, we have

$$\mathbb{E}[\mathbb{1}_{F\cap\{\tau=n\}}\mathbb{1}_{G_\tau} \mid \mathcal{F}_n] = \mathbb{1}_{\{\tau=n\}}\mathbb{1}_F\mathbb{E}[\mathbb{1}_{G_n}]$$

Since $\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau=n\}} = \mathbb{1}_{E^c}$, we have from the linearity and tower property of expectation

$$\mathbb{E}[\mathbb{1}_{F\cap G_\tau}] = \mathbb{E}\left[\mathbb{1}_F\mathbb{1}_{G_\tau}\sum_{n\in\mathbb{N}}\mathbb{1}_{\{\tau=n\}}\right] = \mathbb{E}\left[\sum_{n\in\mathbb{N}}\mathbb{E}[\mathbb{1}_{F\cap\{\tau=n\}}\mathbb{1}_{G_\tau} \mid \mathcal{F}_n]\right] = \mathbb{E}[\mathbb{1}_F\sum_{n\in\mathbb{N}}\mathbb{1}_{\{\tau=n\}}\mathbb{E}[\mathbb{1}_{G_n}]] = \mathbb{E}[\mathbb{1}_F]\mathbb{E}[\mathbb{1}_{G_\tau}].$$

Therefore, the result follows. ☐

Example 2.16. Let $S : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ denote the random walk associated with the *i.i.d.* Bernoulli step size sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ where $\mathbb{E}X_1 = p$. We observe that the kth hitting time to $\{1\}$ by step size sequence X is the first hitting time to $\{k\}$ by random walk S. That is,

$$\tau_S^{\{k\}} = \inf\{n \in \mathbb{N} : S_n = k\} = \tau_S^{\{k-1\}} + \inf\left\{n \in \mathbb{N} : S_{\tau_S^{\{k-1\}}+n} - S_{\tau_S^{\{k-1\}}} = 1\right\}.$$

We recall that $\tau_S^{\{1\}}$ is finite almost surely, and we will show that $\tau_S^{\{k\}}$ is finite almost surely for all $k \in \mathbb{N}$ by induction. By induction hypothesis, $\tau_S^{\{k-1\}}$ is finite almost surely. Then $S_{\tau_S^{\{k-1\}}+n} - S_{\tau_S^{\{k-1\}}} = \sum_{j=1}^n X_{\tau_S^{\{k-1\}}+j}$ is the sum of n *i.i.d.* Bernoulli random variables, and hence has distribution identical to S_n. Further, since X is *i.i.d.* and $\tau_S^{(k-1)}$ is a stopping time, the collection $(X_{\tau_S^{\{k-1\}}+j} : j \in \mathbb{N})$ is independent of the past $(X_j : j \leqslant \tau_S^{\{k-1\}})$. This implies that $\tau_S^{\{k\}} = \tau_S^{\{k-1\}} + \tau^{\{1\}}$, where $\tau^{\{1\}}$ has the identical distribution to $\tau_X^{\{1\}}$ and is independent of $\tau_X^{\{1\}}$. Since the sum of almost surely finite random variables is finite, it follows that $\tau_S^{\{k\}}$ is almost surely finite. Further, we can write

$$\mathbb{E}\tau_S^{\{k\}} = \mathbb{E}\tau_S^{\{k-1\}} + \mathbb{E}\tau_S^{\{1\}} = k\mathbb{E}\tau_S^{\{1\}} = \frac{k}{\mathbb{E}X_1}.$$