Lecture-18: Tractable Random Processes

1 Examples of Tractable Stochastic Processes

In general, it is very difficult to characterize a stochastic process completely in terms of its finite dimensional distribution. However, we have listed few analytically tractable examples below, where we can completely characterize the stochastic process. We will consider the probability space $(\Omega, \mathcal{F}, P)$, and a random process $X : \Omega \to \mathcal{X}^T$ for index set T and state space $\mathcal{X} \subseteq \mathbb{R}$.

1.1 Independent and identically distributed (*i.i.d.*) processes

Definition 1.1 (*i.i.d.* process). A random process $X : \Omega \to \mathcal{X}^T$ is **independent and identically distributed** (*i.i.d.*) with the common distribution $F : \mathbb{R} \to [0,1]$, if for any finite $S \subseteq T$ and a real vector $x_S \in \mathbb{R}^S$, we can write the finite dimensional distribution for this process as

$$F_{X_S}(x_S) = P\left(\cap_{s\in S}\{X_s(\omega) \leqslant x_s\}\right) = \prod_{s\in S} F(x_s).$$

Remark 1. It's easy to verify that all cross moments are independent of time indices. That is, if $0 \in T$ then $X_t = X_0$ in distribution, and we have

$$m_X(t) = \mathbb{E}X_0, \qquad R_X(t,s) = (\mathbb{E}X_0^2)\mathbb{1}_{\{t=s\}} + m_X^2\mathbb{1}_{\{t\neq s\}} \qquad C_X(t,s) = \mathrm{Var}(X_0)\mathbb{1}_{\{t=s\}}.$$

Example 1.2. For an *i.i.d.* Bernoulli process $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with $\mathbb{E}X_1 = p$, the common distribution is $F(x) = \mathbb{1}_{\{x\geqslant 1\}} + (1-p)\mathbb{1}_{[0,1}(x)$. Therefore, defining $n(x_S) \triangleq \sum_{s\in S}\mathbb{1}_{[0,1)}(x_s)$, we get

$$F_{X_S}(x_S) = \begin{cases}(1-p)^{n(x_S)}, & \min\{x_s : s \in S\} \geqslant 0,\\ 0, & \min\{x_s : s\in S\} < 0.\end{cases}$$

1.2 Stationary processes

Definition 1.3 (Stationary process). We consider the index set $T \subseteq \mathbb{R}$. A stochastic process $X : \Omega \to \mathcal{X}^T$ is **stationary** if all finite dimensional distributions are shift invariant. That is, for any finite $S \subseteq T$ and $t \in T$, we have

$$F_{X_S}(x_S) = P(\cap_{s\in S}\{X_s(\omega) \leqslant x_s\}) = P(\cap_{s\in S}\{X_{t+s}(\omega) \leqslant x_s\}) = F_{X_{t+S}}(x_S).$$

Remark 2. That is, for any finite $n \in \mathbb{N}$ and $t \in T$, the random vectors $(X_{s_1}, \dots, X_{s_n})$ and $(X_{t+s_1}, \dots, X_{t+s_n})$ have the identical joint distribution for all $s_1 \leqslant \dots \leqslant s_n$.

Remark 3. In particular, all the cross moments are shift invariant when they exist. For any finite $n \in \mathbb{N}$ $S \triangleq \{s_1, \dots, s_n\} \subseteq T$, we can take shift s_1 and hence for $S' \triangleq \{0, s_2 - s_1, \dots, s_n - s_1\} \subseteq T$, the random vectors X_S and $X_{S'}$ have the identical joint distribution. Therefore, we have $\mathbb{E}\prod_{i\in n} X_{s_i} = \mathbb{E}[X_0 \prod_{i=2}^{n-1} X_{s_i - s_1}]$.

Lemma 1.4. *Any* i.i.d. *process with index set $T \subseteq \mathbb{R}$ is stationary.*

Proof. Let $X : \Omega \to \mathcal{X}^T$ be an *i.i.d.* random process, where $T \subseteq \mathbb{R}$. Then, for any finite index subset $S \subseteq T, t \in T$ and $x_S \in \mathbb{R}^S$, we can write

$$F_{X_S}(x_S) = P(\cap_{s \in S} \{X_s \leqslant x_s\}) = \prod_{s \in S} P\{X_s \leqslant x_s\} = \prod_{s \in S} P\{X_{s+t} \leqslant x_s\} = P(\cap_{s \in S} \{X_{t+s} \leqslant x_s\}) = F_{X_{t+S}}(x_S).$$

First equality follows from the definition, the second from the independence of process X, the third from the identical distribution for the process X. In particular, we have shown that process X is also stationary. □

Definition 1.5. Two processes $X : \Omega \to \mathcal{X}^T$ and $Y : \Omega \to \mathcal{Y}^T$ are jointly stationary if processes X and Y are stationary, and the joint finite dimensional distributions are shift invariant.

Definition 1.6. A **second order** stochastic process X has finite auto-correlation $R_X(t,t) < \infty$ for all indices $t \in T$.

Remark 4. For a second order stochastic process X, we have $X_t \in L^2(\mathcal{F})$ for all times $t \in T$. This implies that $X_t \in L^1(\mathcal{F})$ for all $t \in T$, and hence $m_X(t) = \mathbb{E}X_t < \infty$ for all $t \in T$. Further, from the Cauchy-Schwartz inequality, we obtain that

$$R_X(s,t) = \mathbb{E}X_s X_t \leqslant \sqrt{\mathbb{E}X_s^2 \mathbb{E}X_t^2} = \sqrt{R_X(s,s)R_X(t,t)} < \infty.$$

Therefore, the mean, the auto-correlation, and the auto-covariance functions are well defined and finite.

Remark 5. For a stationary process X, we have $X_t = X_0$ and $(X_t, X_s) = (X_{t-s}, X_0)$ in distribution. Therefore, for a second order stationary process X, we have

$$m_X = \mathbb{E}X_0, \qquad R_X(t,s) = \mathbb{E}X_{t-s}X_0 = R_X(t-s,0), \qquad C_X(t,s) = R_X(t-s,0) - m_X^2 = C_X(t-s,0).$$

Definition 1.7. A random process X is **wide sense stationary** if

1. $m_X(t) = m_X(t+s)$ for all $s, t \in T$, and
2. $R_X(t,s) = R_x(t+u, s+u)$ for all $s, t, u \in T$.

Remark 6. It follows that a second order stationary stochastic process X, is wide sense stationary. A second order wide sense stationary process is not necessarily stationary.

Definition 1.8. Two processes $X : \Omega \to \mathcal{X}^T$ and $Y : \Omega \to \mathcal{Y}^T$ are jointly wide sense stationary, if processes X and Y are wide sense stationary, and $R_{X,Y}(t,s) = R_{X,Y}(t+u, s+u)$, for all $s, t, u \in T$.

Example 1.9 (Gaussian process). Let $X : \Omega \to \mathbb{R}^{\mathbb{R}}$ be a zero-mean continuous-time Gaussian process, defined by its finite dimensional distributions. In particular, for any finite $S \subset \mathbb{R}$, column vector $x_S \in \mathbb{R}^S$, and the covariance matrix $C_{X_S} \triangleq \mathbb{E}X_S X_S^T$, the finite-dimensional density is given by

$$f_{X_S}(x_S) = \frac{1}{(2\pi)^{|S|/2}\sqrt{\det(C_{X_S})}} \exp\left(-\frac{1}{2} x_S^T C_{X_S}^{-1} x_S\right).$$

Theorem 1.10. *A wide sense stationary Gaussian process is stationary.*

Proof. For Gaussian random processes, first and the second moment suffice to get any finite dimensional distribution. Let X be a wide sense stationary Gaussian process and let $S \subseteq \mathbb{R}$ be finite. From the wide sense stationarity of X, we have $\mathbb{E}X_S = m_X[1, \dots, 1]^T$ and

$$C_{X_S}(s,u) = \mathbb{E}(X_s - m_X)(X_u - m_X) = C_X(s-u), \text{ for all } s, u \in S.$$

This means that $C_{X_S} = C_{X_{t+S}}$, and the result follows.

1.3 Markov processes

Definition 1.11. A stochastic process $X : \Omega \to \mathcal{X}^T$ for state space $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set T is **Markov** if conditioned on the present state, future is independent of the past. We denote the history of the process until time t as $\mathcal{F}_t = \sigma(X_s, s \leqslant t)$. That is, for any two indices $u > t$, we have

$$P(\{X_u \leqslant x_u\} \mid \mathcal{F}_t) = P(\{X_u \leqslant x_u\} \mid \sigma(X_t)).$$

Remark 7. For an event $E \in \mathcal{F}$ and a sub-event space $\mathcal{G} \subseteq \mathcal{F}$, the conditional probability $P(E \mid \mathcal{G}) = \mathbb{E}[\mathbb{1}_E \mid \mathcal{G}]$. Therefore, to show Markov property, it suffices to show that

$$\mathbb{E}[\mathbb{1}_F \mathbb{1}_{\{X_u \leqslant x_u\}}] = \mathbb{E}[\mathbb{1}_F \mathbb{E}[\mathbb{1}_{\{X_u \leqslant x_u\}} \mid \sigma(X_t)]], \quad x_u \in \mathbb{R}, u > t, F \in \mathcal{F}_t.$$

Remark 8. We next re-write the Markov property more explicitly for the process X. For all $x, y \in \mathcal{X}$, finite set $S \subseteq T$ such that $\max S < t < u$, and $H_S = \cap_{s \in S} \{X_s \leqslant x_s\} \in \mathcal{F}_t$, we have

$$P(\{X_u \leqslant y\} \mid H_S \cap \{X_t \leqslant x\}) = P(\{X_u \leqslant y\} \mid \{X_t \leqslant x\}).$$

Remark 9. When the state space $\mathcal{X}$ is countable, we can write $H_S = \cap_{s \in S} \{X_s = x_s\}$ and the Markov property can be written as

$$P(\{X_u = y\} \mid H_S \cap \{X_t = x\}) = P(\{X_u = x_u\} \mid \{X_t = x\}).$$

Remark 10. In addition, when the index set is countable, i.e. $T = \mathbb{Z}_+$, then we can take past as $S = \{0, \dots, n-1\}$, present as instant n, and the future as $n+1$. Then, the Markov property can be written as

$$P(\{X_{n+1} = y\} \mid H_{n-1} \cap \{X_n = x\}) = P(\{X_{n+1} = y\} \mid \{X_n = x\}),$$

for all $n \in \mathbb{Z}_+, x, y \in \mathcal{X}$.

We will study this process in detail in coming lectures.

Lemma 1.12. *Any independent process $X : \Omega \to \mathcal{X}^T$ with index set $T \subseteq \mathbb{R}$ is Markov.*

Proof. Let $\mathcal{F}_\bullet$ be the natural filtration of the independent process X. Then for any $u > t$, the random variable X_u is independent of $\mathcal{F}_t$, and hence $P(\{X_u \leqslant x_u\} \mid \mathcal{F}_t) = P(\{X_u \leqslant x_u\} \mid \sigma(X_t)) = F_{X_u}(x_u)$. □

Proof. Alternatively, we observe that for any $F \in \mathcal{F}_t$ and $u > t$, the events F and $\{X_u \leqslant x_u\}$ are independent. Therefore, $\mathbb{E}[\mathbb{1}_F \mathbb{E}[\mathbb{1}_{\{X_u \leqslant x_u\}} \mid \sigma(X_t)]] = \mathbb{E}[\mathbb{1}_F]\mathbb{E}\mathbb{1}_{\{X_u \leqslant x_u\}} = \mathbb{E}[\mathbb{1}_F \mathbb{1}_{\{X_u \leqslant x_u\}}]$. □

Lemma 1.13. *For an independent step size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the associated random walk $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is Markov.*

Proof. Let $\mathcal{F}_\bullet$ be the natural filtration of the independent process X. Then for any $m > n$, the random variable X_m is independent of $\mathcal{F}_n$. Therefore, the difference $S_m - S_n$ is independent of $\mathcal{F}_n$, and S_n is $\mathcal{F}_n$ measurable. We can write $S_m = S_n + S_m - S_n$, and let the distribution of $S_m - S_n$ be G, then $\mathbb{E}\left[\mathbb{1}_{\{S_m - S_n \leqslant s - S_n\}} \mid \mathcal{F}_n\right] = G(s - S_n) = \mathbb{E}\left[\mathbb{1}_{\{S_m - S_n \leqslant s - S_n\}} \mid \sigma(S_n)\right]$. □

Proof. We can show this for countable state space $\mathcal{X}$ such that $\Omega = \cup_{x \in \mathcal{X}} \{S_n = x\}$. From the linearity of conditional expectation and dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{S_m \leqslant s\}} \mid \mathcal{F}_n] &= \sum_{x \in \mathcal{X}} \mathbb{E}[\mathbb{1}_{\{S_m - S_n \leqslant s - x\}} \mathbb{1}_{\{S_n = x\}} \mid \mathcal{F}_n] = \sum_{x \in \mathcal{X}} \mathbb{E}[\mathbb{1}_{\{S_m - S_n \leqslant s - x\}}] \mathbb{1}_{\{S_n = x\}} \\
&= \sum_{x \in \mathcal{X}} \mathbb{E}[\mathbb{1}_{\{S_m - S_n \leqslant s - x\}} \mathbb{1}_{\{S_n = x\}} \mid \sigma(S_n)] = \mathbb{E}[\mathbb{1}_{\{S_m \leqslant s\}} \mid \sigma(S_n)].
\end{aligned}$$

□

1.4 Lévy processes

A right continuous with left limits stochastic process $X : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ with $X_0 = 0$ almost surely, is a **Lévy process** if the following conditions hold.

(L1) The increments are independent. For any instants $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$, the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(L2) The increments are stationary. For any instants $0 \leqslant t_1 < t_2 < \cdots < t_n < \infty$ and time-difference $s > 0$, the random vectors $(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}})$ and $(X_{s+t_2} - X_{s+t_1}, X_{s+t_3} - X_{s+t_2}, \ldots, X_{s+t_n} - X_{s+t_{n-1}})$ are equal in distribution.

(L3) Continuous in probability. For any $\epsilon > 0$ and $t \geqslant 0$ it holds that $\lim_{h \to 0} P\{|X_{t+h} - X_t| > \epsilon\} = 0$.

Example 1.14. Two examples of Lévy processes are Poisson process and Wiener process. The distribution of Poisson process at time t is Poisson with rate λt and the distribution of Wiener process at time t is zero mean Gaussian with variance t.

Example 1.15. For an *i.i.d.* step size sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$, the associated random walk $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ defined by $S_n \triangleq \sum_{i=1}^{n} X_i$ for all $n \in \mathbb{N}$, has stationary and independent increments. Independence of increments of random walk S follows from the independence of step size sequence X. Stationarity of increments of random walk S follows from the identical distribution of step size sequence X.

Further, the random process $\tau_S : \Omega \to \mathbb{N}^{\mathbb{N}}$ defined by $\tau_S^{\{k\}} \triangleq \inf\{n \in \mathbb{N} : S_n = k\}$ for all $k \in \mathbb{N}$, is a random walk with *i.i.d.* step size sequence $Y : \Omega \to \mathbb{N}^{\mathbb{N}}$ defined by $Y_k \triangleq \tau_S^{\{k\}} - \tau_S^{\{k-1\}}$ for all $k \in \mathbb{N}$. Therefore, process τ_S has stationary and independent increments.