Lecture-24: Poisson Processes

1 Simple point processes

Consider the d-dimensional Euclidean space $\mathbb{R}^d$. The collection of Borel measurable subsets $\mathcal{B}(\mathbb{R}^d)$ of the above Euclidean space is generated by sets $B(x) \triangleq \left\{y \in \mathbb{R}^d : y_i \leqslant x_i\right\}$ for $x \in \mathbb{R}^d$.

Definition 1.1. A **simple point process** is a random countable collection of distinct points $S : \Omega \to \mathbb{R}^{d^{\mathbb{N}}}$, such that the distance $\|S_n\| \to \infty$ as $n \to \infty$.

Example 1.2 (Simple point process on the half-line). We can simplify this definition for $d = 1$. In $\mathbb{R}_+$, one can order the points of the process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ to get another process $\tilde{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, such that $\tilde{S}_n = S_{(n)}$ is the nth order statistics of S. That is, $S_{(0)} \triangleq 0$, and $S_{(n)} \triangleq \inf\left\{S_k > S_{(n-1)} : k \in \mathbb{N}\right\}$. such that $S_{(1)} < S_{(2)} < \cdots < S_{(n)} < \ldots$, and $\lim_{n\in\mathbb{N}} S_{(n)} = \infty$. The Borel measurable sets for $\mathbb{R}_+$ are generated by the collection of half-open intervals $\{(0,t] : t \in \mathbb{R}_+\}$.

Point processes can model many interesting physical processes.

1. Arrivals at classrooms, banks, hospital, supermarket, traffic intersections, airports etc.
2. Location of nodes in a network, such as cellular networks, sensor networks, etc.

Definition 1.3. Corresponding to a point process S, we denote the number of points in a set $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$N(A) = \sum_{n\in\mathbb{N}} 1_{\{S_n \in A\}}, \text{ where we have } N(\emptyset) = 0.$$

Then, $N : \Omega \to {\mathbb{Z}_+}^{\mathcal{B}(\mathbb{R}^d)}$ is called a **counting process** for the point process $S : \Omega \to \mathbb{R}^{d^{\mathbb{N}}}$.

Definition 1.4. A counting process is **simple** if the underlying process is simple.

Remark 1. Let $N : \Omega \to {\mathbb{Z}_+}^{\mathcal{B}(\mathcal{X})}$ be the counting process for the point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$.

i. Note that the point process S and the counting process N carry the same information.

ii. The distribution of point process S is completely characterized by the finite dimensional distributions $(N(A_1),\ldots,N(A_k) :$ bounded $A_1,\ldots,A_k \in \mathcal{B})$ for some finite $k \in \mathbb{N}$.

Example 1.5 (Simple point process on the half-line). The number of points in the half-open interval $(0,t]$ is denoted by

$$N(t) \triangleq N((0,t]) = \sum_{n\in\mathbb{N}} 1_{\{S_n\in(0,t]\}}.$$

Since the Borel measurable sets $\mathcal{B}(\mathbb{R}_+)$ are generated by half-open intervals $\{(0,t] : t \in \mathbb{R}_+\}$, we denote the counting process by $N : \Omega \to \mathbb{Z}_+{}^{\mathbb{R}_+}$, where $N(t) = N((0,t])$. For $s < t$, the number of points in interval $(s,t]$ is $N((s,t]) = N((0,t]) - N((0,s]) = N(t) - N(s)$.

Theorem 1.6. *Distribution of a simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is completely determined by void probabilities $(P\{N(A) = 0\} : A \in \mathcal{B}(\mathcal{X}))$.*

Proof. We will show this by induction on the number of points in a bounded set $A \in \mathcal{B}$. We assume that $(B_k \in \mathcal{B}(\mathcal{X}) : k \in \mathbb{N})$ is a sequence of sufficiently small sets partitioning $\mathcal{X}$ such that $N(B_i) \in \{0,1\}$ for all $i \in \mathbb{N}$. We define $A_k \triangleq B_k \cap A$ to write

$$P\{N(A) = 1\} = \sum_{k\in\mathbb{N}} P\{N(A_k) = 1, N(A \setminus A_k) = 0\}.$$

We observe the following set equality $\{N(A_k) = 1, N(A \setminus A_k) = 0\} \cup \{N(A) = 0\} = \{N(A \setminus A_k) = 0\}$, and hence we can write

$$P\{N(A) = 1\} = \sum_{k\in\mathbb{N}} (P\{N(A \setminus A_k) = 0\} - P\{N(A) = 0\}).$$

We assume that $\{N(A) = n\}$ and by the induction hypothesis $P\{N(B) = n-1\}$ can be completely characterized by the void probabilities for all bounded sets $B \in \mathcal{B}$. Then, we can write

$$P\{N(A) = n\} = (1 - \frac{1}{n})P\{N(A) = n-1\} + \frac{1}{n}\sum_{k\in\mathbb{N}} (P\{N(A \setminus A_k) = n-1\} - P\{N(A) = n-1\}).$$

□

Definition 1.7. A non-negative integer valued random variable $N : \Omega \to \mathbb{Z}_+$ is called **Poisson** if for some constant $\lambda > 0$, we have

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}.$$

Remark 2. It is easy to check that $\mathbb{E}N = \mathrm{Var}[N] = \lambda$. Furthermore, the moment generating function $M_N(t) = \mathbb{E}e^{tN} = e^{\lambda(e^t-1)}$ exists for all $t \in \mathbb{R}$.

Corollary 1.8. *A simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has Poisson marginal distribution if and only if void probabilities are exponential.*

Proof. We will show that if void probabilities are exponential, then the marginal distribution is Poisson. Since A_k are very small, we get $e^{\Lambda(A_k)} - 1 \approx \Lambda(A_k)$. Therefore, we write that

$$P\{N(A) = 1\} = \sum_{k\in\mathbb{N}} e^{-\Lambda(A)}(e^{\Lambda(A_k)} - 1) \approx \Lambda(A)e^{-\Lambda(A)}.$$

By inductive hypothesis, let $P\{N(A) = n-1\} = e^{-\Lambda(A)}\frac{\Lambda(A)^{n-1}}{(n-1)!}$ for all bounded $A \in \mathcal{B}(\mathcal{X})$ and $n \in \mathbb{N}$. Then, we can write

$$P\{N(A) = n\} = e^{-\Lambda(A)}\frac{\Lambda(A)^{n-1}}{(n-1)!}(1 - \frac{1}{n} + \frac{1}{n}\sum_{k\in\mathbb{N}}\left(e^{\Lambda(A_k)}\left(1 - \frac{\Lambda(A_k)}{\Lambda(A)}\right)^{n-1} - 1\right).$$

From the fact that $\sum_{k\in\mathbb{N}} \Lambda(A_k) = \Lambda(A)$ and the approximation $e^{\Lambda(A_k)}\left(1 - \frac{\Lambda(A_k)}{\Lambda(A)}\right)^{n-1} - 1 \approx \Lambda(A_k) - (n-1)\frac{\Lambda(A_k)}{\Lambda(A)}$ for sufficiently small A_k, we get the induction step $P\{N(A) = n\} \approx e^{-\Lambda(A)}\frac{\Lambda(A)^n}{n!}$. □

Definition 1.9. A counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has the **completely independence property**, if for any collection of finite disjoint and bounded sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{X})$, the vector $(N(A_1), \dots, N(A_k)) : \Omega \to \mathbb{Z}_+^k$ is independent. That is,

$$P \bigcap_{i=1}^{k} \{N(A_i) = n_i\} = \prod_{i=1}^{k} P\{N(A_i) = n_i\}, \quad n \in \mathbb{Z}_+^k.$$

2 Poisson point process

Remark 3. Recall that $|A| = \int_{x \in A} dx$ is the volume of the set $A \in \mathcal{B}(\mathbb{R}^d)$ and for any such A, the intensity measure of this set is scaled volume

$$\Lambda(A) = \int_{x \in A} \lambda(x) dx,$$

for the intensity density $\lambda : \mathbb{R}^d \to \mathbb{R}_+$. If the intensity density $\lambda(x) = \lambda$ for all $x \in \mathbb{R}^d$, then $\Lambda(A) = \lambda |A|$. In particular for partition $A_1, \dots, A_k$ for a set A, we have $\Lambda(A) = \sum_{i=1}^{k} \Lambda(A_i)$.

Definition 2.1. A simple point process $S : \Omega \to \mathcal{X}^{\mathbb{N}}$ is **Poisson point process**, if the associated counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathcal{X})}$ has complete independence property and the marginal distributions are Poisson.

Definition 2.2. The **intensity measure** $\Lambda : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$ of Poisson process S is defined by $\Lambda(A) \triangleq \mathbb{E}N(A)$ for all bounded $A \in \mathcal{B}(\mathcal{X})$.

Remark 4. That is, for a Poisson process with intensity measure Λ, $k \in \mathbb{Z}_+$, and bounded mutually disjoint sets $A_1, \dots, A_k \in \mathcal{B}(\mathcal{X})$, we have

$$P\{N(A_1) = n_1, \dots, N(A_k) = n_k\} = \prod_{i=1}^{k} \left(e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{n_i!} \right), \quad n \in \mathbb{Z}_+^k.$$

Definition 2.3. If the intensity measure Λ of a Poisson process S satisfies $\Lambda(A) = \lambda |A|$ for all bounded $A \in \mathcal{B}(\mathcal{X})$, then we call S a **homogeneous Poisson point process** and λ is its intensity.

3 Equivalent characterizations

Theorem 3.1 (Equivalences). *Following are equivalent for a simple counting process $N : \Omega \to \mathbb{Z}_+{}^{\mathcal{B}(\mathcal{X})}$.*

i_ Process N is Poisson with locally finite intensity measure Λ.

ii_ For each bounded $A \in \mathcal{B}(\mathcal{X})$, we have $P\{N(A) = 0\} = e^{-\Lambda(A)}$.

iii_ For each bounded $A \in \mathcal{B}(\mathcal{X})$, the number of points $N(A)$ is a Poisson with parameter $\Lambda(A)$.

iv_ Process N has the completely independence property, and $\mathbb{E}N(A) = \Lambda(A)$.

Proof. We will show that $i\_ \implies ii\_ \implies iii\_ \implies iv\_ \implies i\_$.

$i \implies ii\_$ It follows from the definition of Poisson point processes and definition of Poisson random variables.

$ii \implies iii\_$ From Theorem 1.6, we know that void probabilities determine the entire distribution.

$iii \implies iv\_$ We will show this in two steps.

Mean: Since the distribution of random variable $N(A)$ is Poisson, it has mean $\mathbb{E}N(A) = \Lambda(A)$.

CIP: For disjoint and bounded $A_1, \dots, A_k \in \mathcal{B}$ and $A = \cup_{i=1}^k A_i$, we have $N(A) = N(A_1) + \dots N(A_1)$. Taking expectations on both sides, and from the linearity of expectation, we get

$$\Lambda(A) = \Lambda(A_1) + \cdots + \Lambda(A_k).$$

From the number of partitions $n_1 + \cdots + n_k = n$, we can write

$$P\{N(A) = n\} = \sum_{n_1+\cdots+n_k=n} P\{N(A_1) = n_1, \dots, N(A_k) = n_k\}.$$

Using the definition of Poisson distribution, we can write the LHS of the above equation as

$$P\{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!} = \prod_{i=1}^k e^{-\Lambda(A_i)} \frac{(\sum_{i=1}^k \Lambda(A_i))^n}{n!}.$$

Since the expansion of $(a_1 + \cdots + a_k)^n = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\dots,n_k} \prod_{i=1}^k a_i^{n_i}$, we get

$$P\{N(A) = n\} = \frac{1}{n!} \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^k e^{-\Lambda(A_i)} \Lambda(A_i))^{n_i} = \sum_{n_1+\cdots+n_k=n} \prod_{i=1}^k e^{-\Lambda(A_i)} \frac{\Lambda(A_i))^{n_i}}{n_i!}.$$

Equating each term in the summation, we get

$$P\{N(A_1) = n_1, \dots, N(A_k) = n_k\} = \prod_{i=1}^k P\{N(A_i) = n_i\}.$$

$iv \implies i_-$ Since void probabilities describe the entire distribution, it suffices to show that $P\{N(A) = 0\} = e^{-\Lambda(A)}$ for all bounded $A \in \mathcal{B}$.

□

Corollary 3.2 (Poisson process on the half-line). *A random process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ indexed by time $t \in \mathbb{Z}_+$ is the counting process associated with a one-dimensional Poisson process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ having intensity measure Λ iff*

(a) Starting with $N(0) = 0$, the process $N(t)$ takes a non-negative integer value for all $t \in \mathbb{R}_+$;

(b) the increment $N(t+s) - N(t)$ is surely nonnegative for any $s \in \mathbb{R}_+$;

(c) the increments $N(t_1), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$ are independent for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$;

(d) the increment $N(t+s) - N(t)$ is distributed as Poisson random variable with parameter $\Lambda((t, t+s])$.

The Poisson process is homogeneous with intensity λ, iff in addition to conditions $(a), (b), (c)$, the distribution of the increment $N(t+s) - N(t)$ depends on the value $s \in \mathbb{R}_+$ but is independent of $t \in \mathbb{R}_+$. That, is the increments are stationary.

Proof. We have already seen that definition of Poisson processes implies all four conditions. Conditions (a) and (b) imply that N is a simple counting process on the half-line, condition (c) is the complete independence property of the point process, and condition (d) provides the intensity measure. The result follows from the equivalence iv_- in Theorem 3.1. □