Lecture-27: Poisson process on the half-line

1 Simple point processes on the half-line

A stochastic process defined on the half-line $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a **counting process** if

1. $N_0 = 0$, and
2. for each $\omega \in \Omega$, the sample path $N(\omega) : \mathbb{R}_+ \to \mathbb{Z}_+$ is non-decreasing, integer valued, and right continuous function of time $t \in \mathbb{R}_+$.

Each discontinuity of the sample path of the counting process can be thought of as a jump of the process, as shown in Figure 1. A simple counting process has the unit jump size almost surely. General point processes in higher dimension don't have any inter-arrival time interpretation.

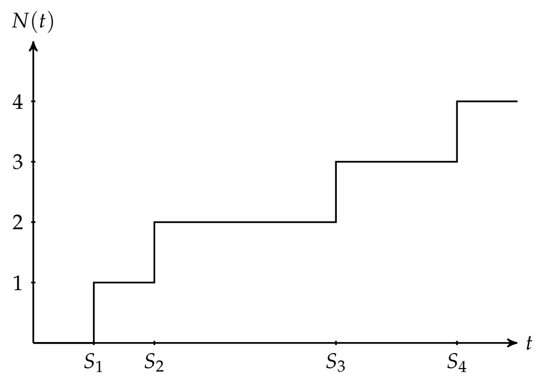

Figure 1: Sample path of a simple counting process.

Lemma 1.1. *A counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ has finitely many jumps in a finite interval $(0, t]$ almost surely.*

The points of discontinuity are also called the arrival instants of the counting process N. The n**th arrival instant** is a random variable denoted $S_n : \Omega \to \mathbb{R}_+$, defined inductively as

$$S_0 \triangleq 0, \qquad S_n \triangleq \inf\{t \geqslant 0 : N_t \geqslant n\}, \quad n \in \mathbb{N}.$$

The **inter arrival time** between $(n-1)$th and nth arrival is denoted by X_n and written as $X_n \triangleq S_n - S_{n-1}$. For a simple point process, we have

$$P\{X_n = 0\} = P\{X_n \leqslant 0\} = 0.$$

Lemma 1.2. *Simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ and arrival process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are inverse processes, i.e.*

$$\{S_n \leqslant t\} = \{N_t \geqslant n\}.$$

Proof. Let $\omega \in \{S_n \leqslant t\}$, then $N_{S_n} = n$ by definition. Since N is a non-decreasing process, we have $N_t \geqslant N_{S_n} = n$. Conversely, let $\omega \in \{N_t \geqslant n\}$, then it follows from definition that $S_n \leqslant t$. □

Corollary 1.3. *For arrival instants $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ associated with a counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ we have $\{S_n \leqslant t, S_{n+1} > t\} = \{N_t = n\}$ for all $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$.*

Proof. It is easy to see that $\{S_{n+1} > t\} = \{S_{n+1} \leqslant t\}^c = \{N_t \geqslant n+1\}^c = \{N_t < n+1\}$. Hence,

$$\{N_t = n\} = \{N_t \geqslant n, N_t < n+1\} = \{S_n \leqslant t, S_{n+1} > t\}.$$

□

Lemma 1.4. *Let $F_n(x)$ be the distribution function for S_n, then $P_n(t) \triangleq P\{N_t = n\} = F_n(t) - F_{n+1}(t)$.*

Proof. It suffices to observe that following is a union of disjoint events,

$$\{S_n \leqslant t\} = \{S_n \leqslant t, S_{n+1} > t\} \cup \{S_n \leqslant t, S_{n+1} \leqslant t\}.$$

□

2 IID exponential inter-arrival times characterization

Proposition 2.1. *The counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with a simple Poisson point process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is Markov.*

Proof. We define the event space $\mathcal{F}_t \triangleq \sigma(N_s : s \leqslant t)$ as the history of the process until time $t \in \mathbb{R}_+$. Then, from the independent increment property of Poisson processes, we have for any historical event $H_s \in \mathcal{F}_s$

$$P(\{N_t = n\} \mid H_s \cap \{N_s = k\}) = P(\{N_t - N_s = n-k\} \mid H_s \cap \{N_s = k\}) = P(\{N_t = n\} \mid \{N_s = k\}).$$

For a homogeneous Poisson point process, the process is homogeneously Markov with $P(\{N_t = n\} \mid \{N_s = k\}) = P\{N(t-s) = n-k\} = e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}$. □

Theorem 2.2. *The counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ associated with a simple Poisson point process $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is strongly Markov.*

Proposition 2.3. *A simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a* ***homogeneous Poisson process*** *with a finite positive rate λ, iff the inter-arrival time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ are* i.i.d. *random variables with an exponential distribution of rate λ.*

Proof. We first assume the *i.i.d.* exponentially distributed inter-arrival times to show that for the simple counting process N and each positive real $t \in \mathbb{R}_+$, the random variable N_t is Poisson with parameter λt, and hence N is homogeneous Poisson with rate λ from the equivalence ii_- in Theorem **??**.

For the converse, let N_t be a simple homogeneous Poisson point process on half-line with rate λ. From equivalence iii_- in Theorem **??**, we obtain for any positive integer t,

$$P\{X_1 > t\} = P\{N_t = 0\} = e^{-\lambda t}.$$

It suffices to show that inter-arrivals time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is *i.i.d.* . We can show that N is Markov process with strong Markov property. Since the sequence of ordered points $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is a sequence of stopping times for the counting process, it follows from the strong Markov property of this process that $(N_{S_n+t} - N_{S_n} : t \geqslant 0)$ is independent of $\sigma(N_s : s \leqslant S_n)$ and hence of S_n and N_{S_n}. Further, we see that

$$X_{n+1} = \inf\{t > 0 : N_{S_n+t} - N_{S_n} = 1\}.$$

It follows that $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is an independent sequence. For homogeneous Poisson point process, we have $N_{S_n+t} - N_{S_n} = N_t$ in distribution, and hence X_{n+1} has same distribution as X_1 for each $n \in \mathbb{N}$.

□

For many proofs regarding Poisson processes, we partition the sample space with the disjoint events $\{N_t = n\}$ for $n \in \mathbb{Z}_+$. We need the following lemma that enables us to do that.

Lemma 2.4. *For any finite time $t > 0$, a Poisson process is finite almost surely.*

Proof. By strong law of large numbers, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}$$

Fix $t > 0$ and we define a sample space subset $M = \{\omega \in \Omega : N(\omega, t) = \infty\}$. For any $\omega \in M$, we have $S_n(\omega) \leqslant t$ for all $n \in \mathbb{N}$. This implies $\limsup_n \frac{S_n}{n} = 0$ and $\omega \notin \left\{\lim_n \frac{S_n}{n} = \frac{1}{\lambda}\right\}$. Hence, the probability measure for set M is zero. □

2.1 Distribution functions

Lemma 2.5. *Moment generating function of arrival times S_n is*

$$M_{S_n}(t) = \mathbb{E}[e^{tS_n}] = \begin{cases} \frac{\lambda^n}{(\lambda - t)^n}, & t < \lambda \\ \infty, & t \geqslant \lambda. \end{cases}$$

Lemma 2.6. *Distribution function of S_n is given by $F_n(t) \triangleq P\{S_n \leqslant t\} = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}$.*

Theorem 2.7. *Density function of S_n is Gamma distributed with parameters n and λ. That is,*

$$f_n(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Theorem 2.8. *For each $t > 0$, the distribution of Poisson process N_t with parameter λ is given by*

$$P\{N_t = n)\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Further, $\mathbb{E}[N_t] = \lambda t$, explaining the rate parameter λ for Poisson process.

Proof. Result follows from density of S_n and recognizing that $P_n(t) = F_n(t) - F_{n+1}(t)$. □

Corollary 2.9. *Distribution of arrival times S_n is*

$$F_n(t) = \sum_{j \geqslant n} P_j(t), \qquad \sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} N_t.$$

Proof. First result follows from the telescopic sum and the second from the following observation.

$$\sum_{n \in \mathbb{N}} F_n(t) = \mathbb{E} \sum_{n \in \mathbb{N}} 1\{N_t \geqslant n\} = \sum_{n \in \mathbb{N}} P\{N_t \geqslant n\} = \mathbb{E} N_t.$$

□

A Poisson process is not a stationary process. That is, the finite dimensional distributions are not shift invariant. This is clear from looking at the first moment $\mathbb{E} N_t = \lambda t$, which is linearly increasing in time.