E2:202 Random Processes **Aug 2020**

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1.1 Cardinality

Definition. A set, A, is *finite* if there exists a bijection $f : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

Definition. A set, A, is *countably infinite* (or *countable*) if there exists a bijection $f : A \to \mathbb{N}$.

Definition. A set that is neither finite nor countable is called *uncountably infinite*.

Let us now look at some examples of these definitions in action:

1. The set of even natural numbers ($2\mathbb{N}$) is countable.
   [Hint: **Think of the mapping** $f : 2\mathbb{N} \to \mathbb{N}$**, given by** $f(i) = i/2, \forall i \in 2\mathbb{N}$**.**]
2. Any subset of a countable set is either finite or countable.
   [Hint: **Can a subset of a countable set be uncountably infinite? Think of what this means for the original countable set.**]
3. The set of integers, $\mathbb{Z}$, is countable.

   Proof. Consider a mapping $f : \mathbb{Z} \to \mathbb{N}$ such that

   $$f(n) = \begin{cases} 2^n & \text{if } n \geq 0, \\ 3^{-n} & \text{if } n < 0. \end{cases}$$

   This map, f, is injective, which implies that the range of this mapping is a *subset* of $\mathbb{N}$. Moreover, since $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{Z}$ can't be finite. Hence, using point 2. above, $\mathbb{Z}$ is *countable.* □

We shall now look at a lemma that will help us prove a surprising result: that $\mathbb{Q}$ has the same cardinality as $\mathbb{N}$.

Lemma (Cartesian Product)**.** $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Proof. Consider the mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(m, n) = 2^m 3^n$, for $m, n \in \mathbb{N}$. Since this mapping is injective, $\mathbb{N} \times \mathbb{N}$ is either finite or countable. But $\mathbb{N} \times \mathbb{N}$ can't possibly be finite (why?). Hence, $\mathbb{N} \times \mathbb{N}$ is countable. □

This leads us to the following simple corollary:

Corollary. A finite cartesian product of countable sets is *countable*.

Proof. This corollary follows from Lemma 1.1, and a simple induction argument. □

Can you now use this corollary to show that $\mathbb{Q}$ is countable?
[Hint: Consider $q \in \mathbb{Q}$ such that $q = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N}$. This defines a mapping from $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$. But we know that $\mathbb{Z} \times \mathbb{N}$ is countable!]

We have now established that $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$ (where the relation $\sim$ is "has the same cardinality as"). We shall now have our first encounter with an *uncountably infinite* set — the set $\mathbb{R}$, of all real numbers.

1.1.1 The Cardinality of $\mathbb{R}$

We shall look at a proof of this by Cantor (1891), for it is an interestng exercise in what is called as "constructive proof". But first, we must take on faith the following fact:

Fact. The set, T, of all infinite sequences of binary digits (0 or 1) is uncountably infinite.

We can then construct an *injective* mapping from T to $\mathbb{R}$, that maps any infinite binary "string" $\underline{s}$ in T to the real number in $\mathbb{R}$ whose decimal (base 10) representation after the decimal point is $\underline{s}$. In other words, the mapping $f : T \to \mathbb{R}$ obeys:

$$f(\underline{s}) = 0.\underline{s},$$

for any $\underline{s} \in T$.

To see an example,

$$\underline{s} = 01000\ldots \xrightarrow{f} r = 0.0100\ldots = \frac{1}{100}$$
$$\underline{s} = 10100\ldots \xrightarrow{f} r = 0.10100\ldots$$

Note that this is an injective mapping. Hence $|\mathbb{R}| \geq |T|$, which implies that $\mathbb{R}$ is uncountable.

Hence, in sum, we have that $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$ and $\mathbb{R}$ is uncountable.

1.2 Review of an Exercise

We now review a question posed in class.

Question. Consider an infinite coin toss experiment. The sample space $\Omega = \{H, T\}^{\mathbb{N}}$. Let $\mathcal{F}$ be the σ-algebra generated by the events $(A_n : n \in [n])$, where

$$A_n := \{\omega \in \Omega : \omega_i = H \text{ for some } i \in [n]\}, \text{for each } n \in \mathbb{N}.$$

In other words, $\mathcal{F} = \sigma(\{A_n : n \in \mathbb{N}\})$.
Define $B_n := \left\{\omega \in \Omega : \omega_n = H,\, \omega^{n-1} = (T, T, \ldots, T)\right\}$. Show that $B_n \in \mathcal{F}$.

Proof. This can be seen simply from the arguments:

$$B_n = A_n \setminus A_{n-1}, \forall n \geq 1 \left(\text{where we define } A_0 \triangleq \phi\right)$$
$$= A_n \cap A_{n-1}^{\complement}.$$

Since $A_n, A_{n-1} \in \mathcal{F}$, it follows that $B_n \in \mathcal{F}$. $\square$

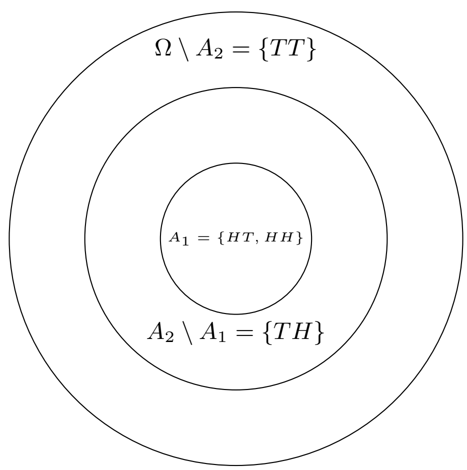
To take a closer look at σ-algebras, consider the sample space $\Omega = \{H, T\}^n$ for some $n \in \mathbb{N}$ (n is finite). Let $\mathcal{F}$ be generated by $(A_i : i \in [n])$, where

$$A_i = \{\omega \in \Omega : \omega_j = H \text{ for some } j \in [i]\}.$$

To see the structure of $\mathcal{F}$ explicitly, we will first set $n = 2$, hence giving us $\Omega = \{H, T\}^2$. We can see that

$$A_1 = \{HT, HH\},$$
$$A_2 = A_1 \cup \{TH\}.$$

Drawing a picture of these nested sets helps:



We can see that $A_1 \subset A_2 \subset \Omega$. Further, B_1 (as defined earlier) is $A_2 \setminus A_1 = \{TH\}$. We can then write down $\mathcal{F}$ explicitly as

$$\mathcal{F} = \{\Omega, \phi, A_1, A_2, \{TT\}, \{TH\}, \{TT, TH\}, \{TT\} \cup A_1\}.$$

Note that the event $C_2 := \{\omega \in \Omega : \omega_2 = H\} = \{HH, TH\}$ does NOT belong to $\mathcal{F}$.

Remark. We can extend this observation to the setting $\Omega = \{H, T\}^{\mathbb{N}}$. $C_n, n \geq 2$ does not belong to $\mathcal{F} = \sigma(\{A_n : n \in \mathbb{N}\})$.

1.3 Algebras and σ-algebras

In class, we saw the the definition of σ-algebras. We will now look at a simpler notion.

Definition (Algebra)**.** Let Ω be a non-empty set. A collection $\mathcal{A}$ of subsets of Ω is called an *algebra* if

(a) $\Omega \in \mathcal{A}$,

(b) for any $A \in \mathcal{A}$, we have $A^c \in \mathcal{A}$ (Closure under complements),

(c) for any $A, B \in \mathcal{A}$, we have $A \cup B \in \mathcal{A}$ (Closure under *finite* unions).

Exercise 1.1. Show that $\forall n \in \mathbb{N}$, $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$, if $A_i \in \mathcal{F}$, for $i \in [n]$.

Exercise 1.2. Given any non-empty set A, such that $A \neq \Omega$, what is the smallest algebra containing A.

Consider the following interesting example.

Example. Let $\Omega = \{r \in \mathbb{Q} : r \in [0,1]\}$ be the set of all rational numbers in the closed interval $[0,1]$.
Let $(A_i \subset \mathbb{Q} : i \in [n])$ for some $n \in \mathbb{N}$ be disjoint sets and let $\mathcal{A} = \uplus_{i=1}^{n} A_i$, where we use $\uplus$ to denote a union of disjoint sets.
Given that for all $i \in [n]$, a_i, b_i belong to the set $\mathbb{Q}$, and $a_i \leq b_i$, and $a_i, b_i \in [0,1]$, the disjoint sets A_i are defined as to be either one of the following:

$$A_i = \begin{cases} \{r \in \mathbb{Q} : a_i < r < b_i\}, & \text{or} \\ \{r \in \mathbb{Q} : a_i \leq r < b_i\}, & \text{or} \\ \{r \in \mathbb{Q} : a_i < r \leq b_i\}, & \text{or} \\ \{r \in \mathbb{Q} : a_i \leq r \leq b_i\}. \end{cases}$$

Claim. $\mathcal{A}$ is an algebra.

Proof. 1. It is easy to see that $\Omega \in \mathcal{A}$ since we can pick $n = 1$, $a_1 = 0$, $b_1 = 1$, and $A_1 = \{r \in \mathbb{Q} : a_1 \leq r \leq b_1\}$.

2. For $A \in \mathcal{A}$, we would like to show that $A^c \in \mathcal{A}$. Can you show this?
Hint: Consider

$$A = \left(\uplus_{i\in[K_1]} (a_i, b_i)\right) \bigcup \left(\uplus_{i\in[K_1+1:K_2]} (a_i, b_i]\right) \bigcup \left(\uplus_{i\in[K_2+1:K_3]} [a_i, b_i)\right) \bigcup \left(\uplus_{i\in[K_3+1:n]} [a_i, b_i]\right),$$

where $K_1 \leq K_2 \leq K_3 \leq n$. Try "ordering" these intervals in an "increasing" fashion and guess what $A^{\complement}$ must look like.

3. Suppose $B = \uplus_{i=1}^{n} C_i$ and $E = \uplus_{j=1}^{m} D_j$, where $C_i, i \in [n], D_j, j \in [m]$ are sets of the form A_i defined earlier. Use the similar "ordering" argument and show that $B \cup E \in \mathcal{A}$.

Hence, $\mathcal{A}$ is indeed an algebra. □

Since Ω is countable, we should be able to write

$$\Omega = \{r_1, r_2, \ldots\}. \quad (\textbf{Why does this hold?})$$

In other words,

$$\Omega = \cup_{i=1}^{n} \{r_i\}$$

where $\{r_i\}$ are singleton sets that are in $\mathcal{A}$. But in this above form, it looks like $\Omega \notin \mathcal{A}$!

This motivates the definition of σ-algebra, to include countable unions.

Example 1.3 (Algebra that is not a σ-algebra). Let $\Omega = \mathbb{R}$. $\mathcal{L}$ is the collection of *finite* disjoint unions of intervals of the form $(-\infty, a], (a, b], (b, \infty), \phi, \mathbb{R}$. This is not a sigma algebra since the *countable* union of $\left(0, \frac{i-1}{i}\right], i \in \mathbb{N}$, is $(0,1)$, which does NOT belong to $\mathcal{L}$.

1.4 Limits of Sets

In class, we saw the following definitions of certain kinds of limits associated with sets $A_n \subset \Omega$ and $A_n \in \mathcal{F}$:

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \text{ and,}$$
$$\liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Now as an exercise, can you prove that both $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ belong to $\mathcal{F}$?
Hint: This crucially depends on the countable unions property of a σ-algebra.

Let us now interpret the two limit sets defined above.

1. Consider
$$A = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m,$$
and let $E_n := \bigcup_{m\geq n} A_m$. We now see that the following sequence of assertions holds:
$$\begin{aligned}\text{If } \omega \in A &\Rightarrow \omega \in E_n, \quad \forall n \in \mathbb{N}\\ &\Rightarrow \omega \in \cup_{m=1}^{\infty} A_m,\ \omega \in \cup_{m=2}^{\infty} A_m, \text{ and so on.}\\ &\Rightarrow \exists n_1 \geq 1 \text{ s.t. } \omega \in A_{n1}, \exists n_1 \geq 2 \text{ s.t. } \omega \in A_{n2}, \text{ and so on.}\\ &\Rightarrow \omega \text{ occurs in } \textbf{infinitely many } A_n.\end{aligned}$$
Hence, the set $A := \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$ is called **A_n infinitely often** or **A_n i.o.**

2. Let us now look at the second limit set,
$$B := \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m,$$
and define $F_n := \bigcap_{m\geq n} A_n$.
$$\begin{aligned}\text{If } \omega \in B &\Rightarrow \omega \text{ belongs to at least one of the } F_n s\\ &\Rightarrow \exists n_0 \in \mathbb{N} \text{ s.t. } \omega \in F_{n_0}\\ &\Rightarrow \exists n_0 \text{ s.t. } \omega \in \cap_{m\geq n_0} A_m\\ &\Rightarrow \omega \text{ occurs in all } A_m\text{s beyond a fixed } n_0 \in \mathbb{N}.\end{aligned}$$
This leads us to understand that ω occurs in **all but finitely many** A_ns. Hence B is called **all but finitely many A_ns** set.

We will see more about the sets A and B as we progress in the course.

1.5 Continuity of Probability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

Recall that if f is a continuous function over the reals, i.e., $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for any sequence of real numbers $\{x_n\}_{n\geq 1}$ s.t. $\lim\limits_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x)$$

Remark. A quick point to note about the limits of real sequences is that when we say that $\lim_{n\to\infty} x_n = x$, we mean that:

$$\forall \epsilon > 0, \text{ there exists an } N \equiv N(\epsilon) \text{ such that } |x_n - x| \leq \epsilon, \quad \forall n \geq N(\epsilon).$$

This means that, beyond a certain N, which could depend on ϵ, the x_ns are at worst ϵ-far away from the limit x.

It is in this spirit that we saw the notion of *continuity of probability.*

Here, if we have an increasing sequence of sets $(A_n \in \mathcal{F})_{n\in\mathbb{N}}$, i.e., $A_n \subseteq A_{n+1} \subseteq A_{n+2} \subseteq \ldots$, whose limit is $A = \bigcup\limits_{n\in\mathbb{N}} A_n$, then

$$\mathbb{P}(A) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Exercise 1.4. For the case when A_ns are increasing, try showing that $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$. [Hint: We know that $\liminf_{n\to\infty} A_n = \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{m\geq n} A_m = \bigcup\limits_{n\in\mathbb{N}} A_n = A$. Now use the fact that $\limsup_{n\to\infty} A_n \subset \bigcup\limits_{k\in\mathbb{N}} A_k = A$, to show that $\limsup_{n\to\infty} A_n = A$.]

In class, we had seen an equivalent notion for decreasing sets, i.e., if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, then

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Let us see an example of this in action

Example. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbb{P}([a,b]) = \mathbb{P}([a,b]) = \mathbb{P}((a,b]) = \mathbb{P}([a,b)) = b - a$ for $a, b \in [0,1], a \leq b$ (this is called the *Lebesgue measure*).
Let $B_n = \left[0, \frac{n}{n+1}\right], n \in \mathbb{N}$. What is $\lim_{n\to\infty} \mathbb{P}(B_n)$?

Note. Any question in probability must begin with a complete description of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Solution: Since the B_ns form an increasing sequence of sets,

$$\lim_{n\to\infty} B_n = \bigcup_{n\in\mathbb{N}} B_n.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(B_n) &= \mathbb{P}\left(\lim_{n\to\infty} B_n\right) \text{ (by the continuity of probability)}\\ &= \mathbb{P}([0,1))\\ &= 1.\end{aligned}$$

Likewise, let $C_n = \left[0, \frac{1}{n}\right)$, for $n \in \mathbb{N}$. This is a decreasing sequence of sets. Then,

$$\lim_{n \to \infty} \mathbb{P}\left(C_n\right) = \mathbb{P}\left(\lim_{n \to \infty} C_n\right) = 0,$$

since the singleton has *measure* zero, i.e., $\mathbb{P}\left(\{0\}\right) = 0$.

Exercise 1.5 (Supplementary Exercise on σ-algebras)**.** Let E, Ω be non-empty sets. For a function $f : \Omega \to E$ and $B \subseteq E$, we can define the pre-image of set B for the map f as

$$f^{-1}\left(B\right) \coloneqq \left\{\omega \in \Omega : f\left(\omega\right) \in B\right\}.$$

Suppose $\mathcal{E}$ is a σ-algebra of subsets of E. Then, show that the following collection of sets is a σ-algebra of subsets of Ω:

$$\mathcal{F} \coloneqq \left\{A \subseteq \Omega : A = f^{-1}\left(B\right), B \in \mathcal{E}\right\}.$$