E2:202 Random Processes **Oct. 2020**

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2.1 Independence and Conditional Probability

Let us first recall the definition of independent events.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space [The mandatory preamble]

Definition (Independence of Events)**.** A family of events $\{A_i : i \in I\}$ s.t. $A_i \in \mathcal{F}, \forall i \in I$ is said to be *independent* if for any finite set $J \subseteq I$,

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i).$$

Further, we had seen the following definition of conditional independence with respect to a given event $C \in \mathcal{F}$:

Definition (Conditional Independence of Events)**.** A family of events $\{A_i \in \mathcal{F} : i \in I\}$ is said to be conditionally independent given an event $C \in \mathcal{F}$ such that $\mathbb{P}(C) > 0$, if for any finite set $J \subseteq I$,

$$P\left(\bigcap_{i \in J} A_i \middle| C\right) = \prod_{i \in J} \mathbb{P}(A_i | C)$$

Let us take a look at some examples of these in action:

Example (Independence $\not\Rightarrow$ Conditional Independence)**.** Let us consider the roll of two dice. Then, $\Omega = \{1, 2, \ldots, 6\}^2, \mathcal{F} = 2^\Omega, \mathbb{P}(\{(i,j)\}) = 1/36 \, \forall \, (i,j) \in \Omega$.

- Let A be the event that the first roll results in a 3.
- Let B be the event that the second roll results in a 1.
- Let C be the event that the sum of the two rolls is *even*.

From the definition of the probability measure on the event space, it is immediate that A, B are independent events, i.e.,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

However, it is NOT true that A, B are conditionally independent given C. To see this,

$$\mathbb{P}(A \cap B \mid C) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(C)}$$
$$\frac{\mathbb{P}(A)\,\mathbb{P}(B)}{\mathbb{P}(C)} = \frac{1/36}{1/2} = \frac{1}{18}$$

On the other hand,

$$\mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C) = \frac{\mathbb{P}(A \cap C)\,\mathbb{P}(B \cap C)}{(\mathbb{P}(C))^2}$$
$$= \frac{\frac{1}{12} \cdot \frac{1}{12}}{\left(\frac{1}{2}\right)^2} = \frac{1}{36} \neq \mathbb{P}(A \cap B \mid C).$$

To see an example where this conditional independence does not imply independence of events, consider the experiment designed as follows:

Example (Conditional Independence $\not\Rightarrow$ Independence)**.** I have two coins C_1, C_2 where C_1 is fair (unbiased) and C_2 shows heads with probability 1. I choose a coin uniformly at random from C_1 and C_2 and toss it twice.
Question: What are $\Omega, \mathcal{F}$ and $\mathbb{P}$, here? Is it that $\Omega = \{H, T\}^2$ and $\mathcal{F} = 2^{\Omega}$? Or does this lead to some difficulty in writing down the probability measure?
Define the following events:

- $A := \{\text{First toss results in a } H\}$
- $B := \{\text{Second toss results in a } H\}$
- $C := \{\text{Coin } C_1 \text{ is selected}\}$

It can be easily be seen that $\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$. However, it must be noted that in this case, events A and B are not independent. To see this, carry out the following computations in a simple exercise:

$$\mathbb{P}(A) = 3/4;\ \mathbb{P}(B) = 3/4;\ \mathbb{P}(A \cap B) = 5/8 \neq \mathbb{P}(A)\,\mathbb{P}(B) = 9/16.$$

The next example is based on a statistical model known as "Polya's Urn Model". This finds applicaton in population genetics, image recognition, and linguistic analysis.

Example. An urn contains r red balls and b blue balls. A ball is chosen at random from the urn, its colour is noted, and it is returned back to the urn along with d more balls of the same colour. This is repeated indefinitely.
Let us assume a suitably defined $(\Omega, \mathcal{F}, \mathbb{P})$.

1. What is the probability that the second ball is blue?
   Solution: Let B_2 be the event that the second ball drawn is blue, and let B_1be the even that the first ball drawn is blue. By an application of the *law of total probability*,

$$\mathbb{P}(B_1) = \mathbb{P}(B_2 \mid B_1)\,\mathbb{P}(B_1) + P\left(B_2 \mid B_1^{\complement}\right)\mathbb{P}\left(B_1^{\complement}\right)$$
$$= \left(\frac{b}{b+r+d}\right)\left(\frac{r}{b+r}\right) + \left(\frac{b+d}{b+r+d}\right) + \left(\frac{b}{b+r}\right)$$
$$= \frac{b}{b+r} \quad \textbf{[This is not dependent on d!]}$$

Remark. In general, let B_n, for $n \geq 1$, $n \in \mathbb{N}$ be the even that the n^{th} ball drawn is blue. Using some sophisticated analysis, it can be shown that

$$\mathbb{P}(B_n) = \mathbb{P}(B_1) = \frac{b}{b+r} \quad \forall n \geq 1,\, n \in \mathbb{N}.$$

2. Find the probability that the first ball is blue given thath the n subsequent balls drawn are all blue. Find the limit of this probability as n tends to ∞.
Solution: Let us make use of notation from the previous remark. Now,

$$\begin{aligned}\mathbb{P}(B_1 \mid B_2 \cap B_3 \cap \dots B_n) &= \frac{\mathbb{P}(B_1 \cap B_2 \cap B_3 \cap \dots B_n)}{\mathbb{P}(B_2 \cap B_3 \cap \dots B_n)} \\ &= \frac{\left(\frac{b}{b+r}\right)\left(\frac{b+d}{b+r+d}\right)\cdots\left(\frac{b+nd}{b+r+nd}\right)}{\left(\frac{b}{b+r}\right)\left(\frac{b+d}{b+r+d}\right)\cdots\left(\frac{b+(n-1)d}{b+r+(n-1)d}\right)\cdot 1} \qquad = \frac{b+nd}{b+r+nd}.\end{aligned}$$

Thus, we have that $\lim_{n\to\infty} \mathbb{P}(B_1 \mid B_2 \cap B_3 \cap \ldots \cap B_n) = 1$.

Exercise 2.1. Give a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{A_n\}_{n\geq 1} \in \mathcal{F}$, and $\{B_n\}_{n\geq 1} \in \mathcal{F}$ such that $A_1 \subseteq A_2 \subseteq \ldots$, and $B_1 \subseteq B_2 \subseteq \ldots$, and $B = \bigcup_{n\in\mathbb{N}} B_n$ $A = \bigcup_{n\in\mathbb{N}} A_n$. Let $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0\, \forall n \in \mathbb{N}$. Show that

(a) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B) = \mathbb{P}(A \mid B)$.

(b) $\lim_{n\to\infty} \mathbb{P}(A \mid B_n) = \mathbb{P}(A \mid B)$.

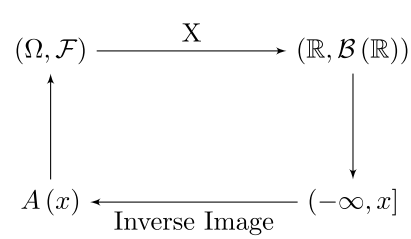
2.2 Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

Definition. A random variable $X : \Omega \to \mathbb{R}$ is a real-valued function that maps elements in Ω to $\mathbb{R}$ such that for each $x \in \mathbb{R}$, the event

$$A(x) \coloneqq \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}.$$

A picture helps cement this idea:



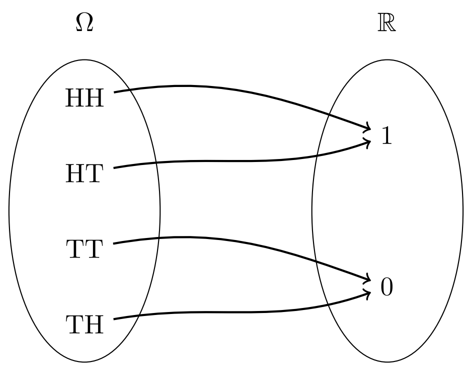
In other words, in order for X to be an $\mathcal{F}$-measurable RV, the inverse image (this is a set inverse!) of $(-\infty, x]$, called $A(x) \subset \Omega$, must belong to $\mathcal{F}$.

Let us look at some examples now.

Example. 1. $\Omega = \{H, T\}^2$, $\mathcal{F} = 2^{\Omega}$
$X : \Omega \to \mathbb{R}$ is given by:

- $X(HH) = 1$,
- $X(HT) = 1$,
- $X(TH) = 0$, and
- $X(TT) = 0$.

X is the RV "heads on the first toss".



We then have

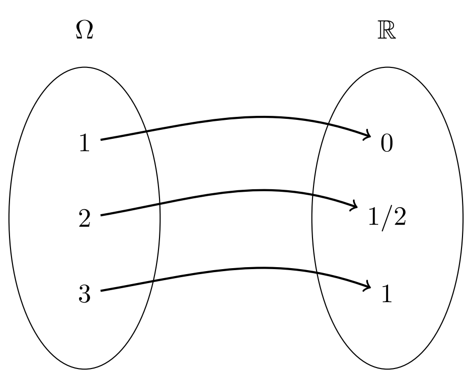
$$X^{-1}(\mathbb{R}) = \Omega, \text{ and}$$
$$X^{-1}((-\infty, x]) = \begin{cases} \phi, \text{ if } x < 0, \\ \{HH, HT\}, \text{ if } 0 \leq x < 1, \\ \Omega, \text{ if } x \geq 1. \end{cases}$$

Note that X is an $\mathcal{F}$-measurable RV

$\Omega = 1, 2, 3, \mathcal{F} = \Omega, \phi, 1, 2, 3$
We define X as follows

$$X(1) = 0, \quad X(2) = 1/2, \quad X(3) = 1.$$

Pictorially, the mapping X is given below:



Now,

$$X^{-1}(\mathbb{R}) = \Omega, \text{ and}$$

$$X^{-1}((-\infty, x]) = \begin{cases} \phi \ x < 0, \\ \{1\} \ \ 0 \le x < 1/2, \\ \{1,2\}, 1/2 \le x < 1, \\ \Omega, x \ge 1. \end{cases}$$

But note that $X^{-1}((-\infty, 2/3]) = \{1,2\} \notin \mathcal{F}$.
Hence X is <u>not</u> $\mathcal{F}$-measurable.

Example (Constructing RVs from σ-algebras)**.** Let $\Omega = \{1,2,3\}$

1. $\mathcal{F} = \{\Omega, \phi\}$. Here, the only $\mathcal{F}$-measurable RVs are constant RVs, i.e.,

$$X(\omega) = c, \ \forall \omega \in \Omega, \ c \in \mathbb{R}$$

2. $\mathcal{F} = \{\Omega, \phi, \{1\}\,\{2,3\}\}$.
   Here, the $\mathcal{F}$-measurable RVs look like

$$X(\omega) = \begin{cases} c_1, \text{ if } \omega = 1, \\ c_2, \text{ if } \omega \in \{2,3\}. \end{cases} \qquad \text{for } c_1, c_2 \in \mathbb{R}.$$

2.2.1 Types of Random Variables, CDF, PMFs, and PDFs

Recall that the cumulative distribution function $F : \mathbb{R} \to [0,1]$ satisfies

(a) for any $x, y \in \mathbb{R}$ s.t. $x \le y$, $F(x) \le F(y)$,

(b) $F(\cdot)$ is right-continuous,

(c) $\lim_{x\to\infty} F(x) = 1$, and $\lim_{x\to-\infty} F(x) = 0$.

This definition also happens to the sufficient conditions for any function F to be a CDF.

2.2.1.1 Discrete Random Variables

1. Bernoulli Random Variables: $X \sim \text{Ber}(p)$.
   In the above, read $\sim$ as "drawn according to". p is a parameter of the distribution.

$$X : \Omega \to \{0,1\}, \text{ such that } F_X(x) = \begin{cases} 0, \ x < 0, \\ 1-p, \ 0 \le x < 1, \\ 1, \ x \ge 1. \end{cases}$$

   <u>Example:</u> The outcome of a coin toss can be thought of as a Bernoulli RV, with suitably defined $p \in [0,1]$.

2. Poisson Random Variable: $X \sim \text{Poi}(\lambda)$, if $X : \Omega \to \mathbb{N} \cup \{0\}$ such that

$$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x \in \mathbb{N} \cup \{0\},\ \text{and}$$
$$F_X(x) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}, x \in \mathbb{N} \cup \{0\}.$$

Question: Why are we writing $P_X(x)$? Is it enough if we define the probability measure over such singletons?
Example: The arrival process of customers in a bank can be modelled as a Poi(λ) random variable.

3. Geometric random variables: $X \sim \text{Geo}(p)$, if $X : \Omega \to \mathbb{N}$, and

$$P_X(x) = p(1-p)^{x-1}, x \in \mathbb{N}$$
$$F_X(x) = \sum_{k=1}^{\infty} p(1-p)^{k-1}, x \in \mathbb{N}.$$

Exercise 2.2. Show that the RV $X \sim \text{Geo}(p)$ satisfies the memoryless property, i.e.,

$$\mathbb{P}\{X > k + n \mid X > n\} \text{ does not depend on } n.$$

2.2.1.2 Continuous Random Variables

Recall that we can specify a continuous random variable X by its density (probability density function, pdf), if it exists.

1. Uniform Random Variable: $X \sim \text{Unif}([a, b])$ if $X : \Omega \to [a, b]$, and

$$f_X(x) = \begin{cases} \frac{1}{b-a}, x \in [a, b], \\ 0, o.w. \end{cases}$$

2. Exponential Random Variable: $X \sim \text{Exp}(\lambda)$ if $X : \Omega \to [0, \infty)$ and

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x},\ x \geq 0, \\ 0,\ \text{o.}w. \end{cases}$$

Exercise 2.3. Show that the memoryless property holds for an Exp(λ) random variable X:

$$\mathbb{P}\{X > t + s \mid X > s\} = \mathbb{P}\{X > t\} = e^{-\lambda t}.$$

3. Normal Random Variables: $X \sim \mathcal{N}(\mu, \sigma^2)$ if $X : \Omega \to \mathbb{R}$ and

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), x \in \mathbb{R}.$$

Exercise 2.4. 1. Suppose that a coin of bias p (probability of a head is $p \in [0, 1]$) is tossed once. Define Ω and the largest possible $\mathcal{F}$. Consider the events:

$$A = \{\text{The coin shows up heads}\},$$
$$B = \{\text{The coin shows up tails}\}.$$

Are events A and B independent?

2. Let us modify the experiment as follows.
   Let N be the RV correspnding to a random number of tosses of the coin. Further, let $N \sim \text{Poi}(\lambda)$. Now, let X denote the number of heads in N (random) tosses, and let Y denote the number of tails. Show (by a low of total probability argument) that:
   (a) $\mathbb{P}(\{X =\}) = \frac{(\lambda p)e^{-\lambda p}}{x!}$, $x \in \mathbb{N}\{0\}$ [Hint: **Try conditioning on** $\{N = n\}$].
   (b) $\mathbb{P}(\{Y = y\}) = \frac{\lambda(1-p)^y e^{-\lambda(1-p)}}{y!}$, $y \in \mathbb{N} \cup \{0\}$.
   (c) What is $\mathbb{P}(\{X = x\} \cap \{Y = y\})$?
   [Hint: **Notice that** $\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(\{X = x\} \cap \{Y = y\} \cap \{N = x + y\})$].

Exercise 2.5. Consider a continuous RV X with $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R})$. Show that if X and $-X$ have the same CDF, then

$$f_X(x) = f_X(-x), \ \forall x \in \mathbb{R}.$$

Exercise 2.6. Consider the Normal RV $Y \sim \mathcal{N}(\mu, \sigma^2)$. Can you write the density function of $Z = aY + b$? Hint: **Start from the CDF of Z and massage it to obtain a CDF of Y.**